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Date	Time	Hours	Type
Tue 25th May	09:00 – 12:00	3	Lessons
Wed 26th May	09:00 – 13:00	4	Exercise - Classwork
Fri 28th May	09:00 – 12:00	3	Lessons
Thu 3rd June	09:00 – 12:00	3	Lessons
Fri 4th June	09:00 – 12:00	3	Exercise - Classwork

Syllabus and Slides

http://polysense.poliba.it/index.php/hollow-core-waveguide-devices/

TEXTBOOKS

C. R. Pollock and M. Lipson – **Integrated Photonics**, Springer.
J. A. Harrington – **Infrared Fibers and Their Applications**, SPIE Press.

SYLLABUS

1. Step-Index Circular Waveguides

1.1. Step-Index Profile. **1.2.** Scalar Helmholtz Equation. **1.3.** Wave Equation in Cylindrical Coordinates. **1.4.** Solution of the Wave Equation for Ez. **1.5.** Field Distribution in the Step-Index Fiber. **1.6.** Boundary Conditions for the Step-Index Waveguide. **1.7.** Transverse Electric and Transverse Magnetic Modes. **Exercise.** **1.8.** The Hybrid Modes. **1.9.** The Linearly Polarized Modes. **1.10.** The Normalized Frequency and Cutoff. **1.11.** Total Number of Modes in a Step-Index Waveguide.

2. Hollow Core Waveguides

2.1. Basic Characteristics. **2.2.** Attenuated Total Internal Reflection. **2.3.** Metallic/Dielectric Coating. 2.3.1. Selection of Metal. 2.3.2. Selection of Dielectric. **2.4.** Propagation Losses. 2.4.1. Metallic Hollow Core Waveguides. 2.4.2. Metallic/Dielectric Hollow Core Waveguides. **2.5.** The Fundamental Hybrid HE_{11} Mode. **2.6.** Launch Conditions and Mode Coupling. **Exercise**

INFO

CHAPTER 1

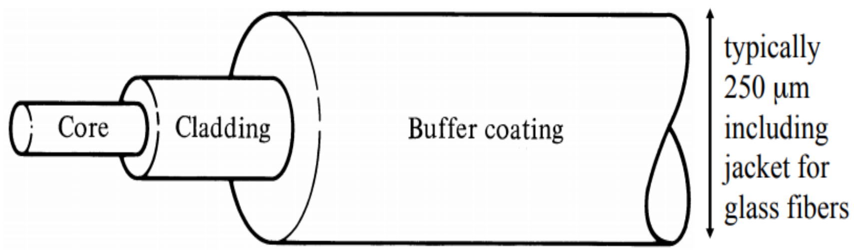
STEP-INDEX CIRCULAR WAVEGUIDES

1.1 STEP-INDEX PROFILE

A circular waveguide consists of two concentric dielectric cylinders, the core and the cladding..

The *core* has a refractive index $n(r)$ and a bore radius a.

The *cladding* has a refractive index n_2 with $n\ (r) > \ n_2$ and a radius $> a$.



The cladding layer is usually covered with a plastic (buffer) coating to protect the fiber from environmental hazards and abrasion.

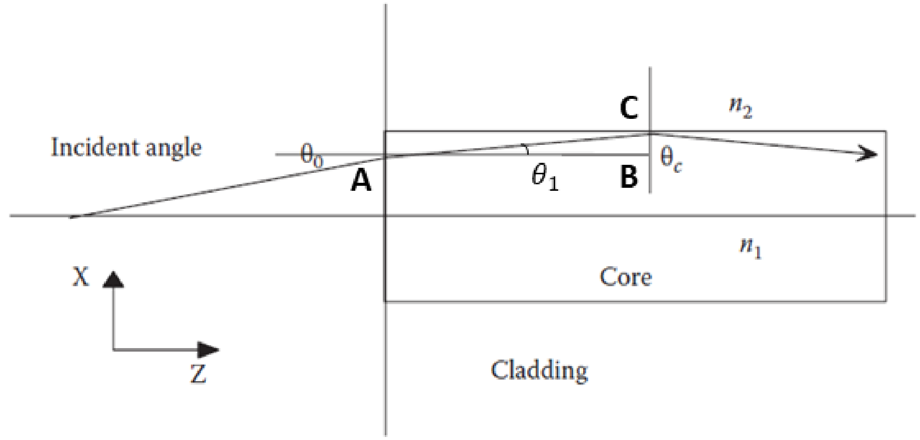
The principle of light propagation through a fiber (guided light) is based on total internal reflection at the *core-cladding* interface, caused the difference in refractive indices of core and cladding.

Let us assume that the light is guided inside the core by total reflection at the core-cladding interface, what is the **acceptance angle** within which the light waves can be launched into the fiber in order to have total reflection?

1.1 STEP-INDEX PROFILE

The concept of acceptance angle is based on ray tracing and refraction.

Consider the optical waveguide shown in the picture, where a high index layer with index n_1 (the core), is surrounded symmetrically by a lower index medium n_2 (the cladding)



We want to explore how light (in the form of rays) can couple into the end of such a structure.

A ray is shown entering the edge of the waveguide with an angle θ_0, where we assume the index of refraction corresponds to that of air (essentially $n_{air} = 1$)

The entering ray is refracted according to Snell's law $sen\theta_0 = n_1 sen\theta_1$

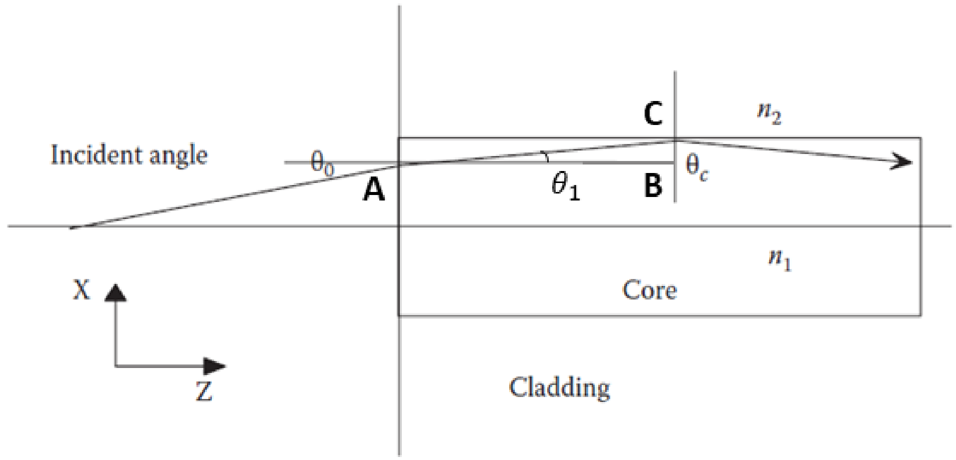
By considering the right triangle ABC in the Picture:

$$\theta_c = 90° - \theta_1$$

By substituting this expression in the Snell's law:

$$sen\theta_0 = n_1 cos\theta_c$$

1.1 STEP-INDEX PROFILE



Being $\theta_c < 180°$, then $cos\theta_c = \sqrt{1 - sen^2\theta_c}$, and so:

$$sen\theta_0 = n_1\sqrt{1 - sen^2\theta_c}$$

$$\boxed{sen\theta_0 = n_1 cos\theta_c}$$

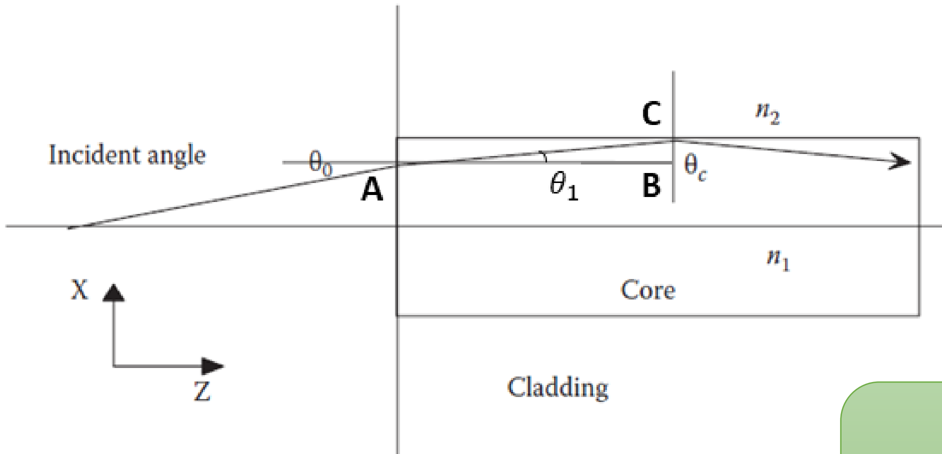
If θ_c is assumed to be the critical angle for total internal reflection at the *core-cladding* interface, we can apply Snell's law at the *core-cladding* interface:

$$n_1 sen\theta_c = n_2 sen90° = n_2$$

Then:

$$sen\theta_c = \frac{n_2}{n_1}$$

1.1 STEP-INDEX PROFILE



$$sen\theta_c = \frac{n_2}{n_1}$$

$$sen\theta_0 = n_1\sqrt{1 - sen^2\theta_c}$$

By combining the two expressions:

$$sen\theta_0 = n_1\sqrt{1 - \frac{{n_2}^2}{{n_1}^2}} = \sqrt{{n_1}^2 - {n_2}^2}$$

The numerical aperture is defined as the sine of the acceptance angle θ_0 that gives total internal reflection at the *core-cladding* interface :

$$NA = \sqrt{{n_1}^2 - {n_2}^2}$$

1.1 STEP-INDEX PROFILE

The refractive index of the core can be not costant along the radial dimension: in this case, $n_1$can be expressed as a function of the fiber radius r, con $0 \leq r \leq a$.

Thus, we have two different cases:

- **Step-Index Profile**. The refractive index profile is characterized by a uniform refractive index within the core and a sharp decrease in refractive index at the core-cladding interface so that the cladding is of a lower refractive index.

$$n_1(r) = n_1$$

- **Graded-Index Profile.** The core has a refractive index that decreases with increasing radial distance from the optical axis of the fiber. In this case, $n_1(r)$ is usually approximated with the following relation:

$$n_1(r) = n_1 + (n_2 - n_1)\left(\frac{r}{a}\right)^{\alpha}$$

If $\alpha = 1$, $n_1(r)$ linearly varies within the core.
If $\alpha = 2$, $n_1(r)$ varies quadratically within the core

1.2 SCALAR HELMHOTZ EQUATION

To find the modes of the circular step-index fiber, we must solve the Maxwell's wave equations in cylindrical coordinates.

The modes of the cylindrical structure are more abstract than those of the planar structure.

Not only are they circular in symmetry which will require a more complicated solution to the wave equation, but they are two dimensional, so there will be two mode numbers.

The wave equation for guided light within a fiber can be obtained by solving Maxwell's equations, with imposition of reasonable boundary conditions related to the geometry of the medium.

Before recall the Maxwell's equations, let us suppose:

- no charge sources ($\rho = 0$ e $j = 0$);
- linear medium (the electric permittivity ϵ as well as the magnetic permeability μ are nondependent on the electric field $\vec{E}$ and the magnetic field $\vec{H}$, respectively);
- isotropic medium.

1.2 SCALAR HELMHOTZ EQUATION

With these conditions superimposed, the Maxwell's equations are:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{D} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

where $\vec{E}$ e $\vec{H}$ are the electric and magnetic field, respectively, and $\vec{B} = \mu \vec{H}$ and $\vec{D} = \epsilon \vec{E}$.

$\vec{D}$ is known as electric displacement field and $\vec{H}$ as auxiliary magnetic field.

These equations describe the propagation of the electric field and the magnetic field over time and space.

The equations are strongly coupled to each other.

Let us combine Maxwell's equations to obtain a second order differential equation for a single field ($\vec{E}$ e $\vec{H}$).

1.2 SCALAR HELMHOTZ EQUATION

Let us take the curl of both sides of first Maxwell's equation:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = \vec{\nabla} \times \left(-\frac{\partial \mu \vec{H}}{\partial t}\right)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{B} = \mu \vec{H}$$

If μ is not time-dependent:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu \left(\vec{\nabla} \times \frac{\partial \vec{H}}{\partial t}\right)$$

Since $\vec{H}$ is a continous function, we can commutate the curl with the partial time derivative:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H})$$

Now we can use the Maxwell's equation for the curl of magnetic field $\vec{\nabla} \times \vec{H}$:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu \frac{\partial}{\partial t} \frac{\partial \vec{D}}{\partial t} = -\mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

1.2 SCALAR HELMHOTZ EQUATION

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = -\mu \frac{\partial}{\partial t} \frac{\partial \vec{D}}{\partial t} = -\mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Let us the following vector identity for the first side, which is true for any vector:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

where ∇^2 is the Laplace operator.

In a Cartesian coordinate system, the Laplacian is given by the sum of second partial derivatives of the function with respect to each independent variable:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

If ϵ is constant, by using the Maxwell's equation $\vec{\nabla} \cdot \vec{D} = 0$, we can assume that the divergence of the electric field is zero $\vec{\nabla} \cdot \vec{E} = 0$.

Thus: $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = -\nabla^2 \vec{E}$

By inserting this last expression in the first equation on the top the slide we can obtain the differential omogeneuos equation for the electric field $\vec{E}$.

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

1.2 SCALAR HELMHOTZ EQUATION

Starting from the other Maxwell's equation $\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$, with same steps as before we can obtain a second-order differential omogeneous equation for the magnetic field:

$$\nabla^2 \vec{H} - \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

Let us suppose separation of space and time variables of $\vec{E}$ and assume a time-harmonic dependence with an angular frequency ω

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}) e^{i\omega t}$$

By imposing $\vec{E}(\vec{r}, t)$ as a solution of the omogeneous equation:

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$e^{i\omega t} \nabla^2 \vec{E}(\vec{r}) + \mu\epsilon\omega^2 \vec{E}(\vec{r}) e^{i\omega t} = 0$$

In an homogeneuos medium, the wavevector k is related to the refractive index n by the relation:

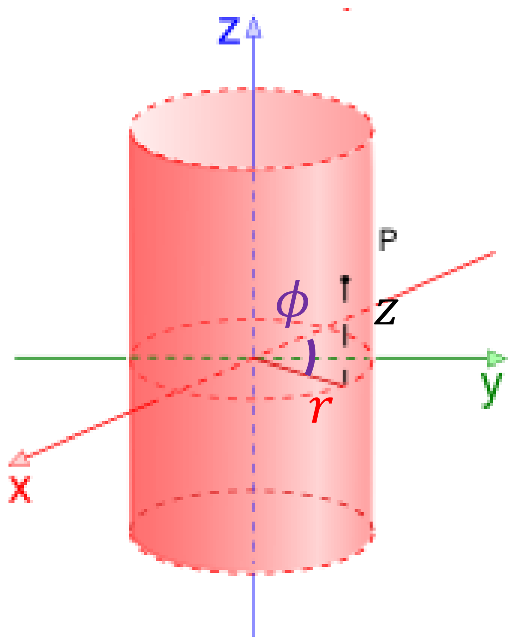
$$k = \omega\sqrt{\mu\epsilon} = k_0 n$$ where k_0 is the wavevector in vacuum

Finally, we obtain the scalar Helmhotz equation for $\vec{E}$:

$$\nabla^2 \vec{E}(\vec{r}) + k_o^2 n^2 \vec{E}(\vec{r}) = 0$$

1.3 WAVE EQUATION IN CYLINDRICAL COORDINATES

To solve the scalar Helmhotz equation in a cylindrical waveguide, we must write this equation in cylindrical coordinates.



$$\nabla^2\vec{E}(\vec{r}) + k_o^2 n^2 \vec{E}(\vec{r}) = 0$$

The electric field is a vector, and there are three components, each of which is a function of r, ϕ e z

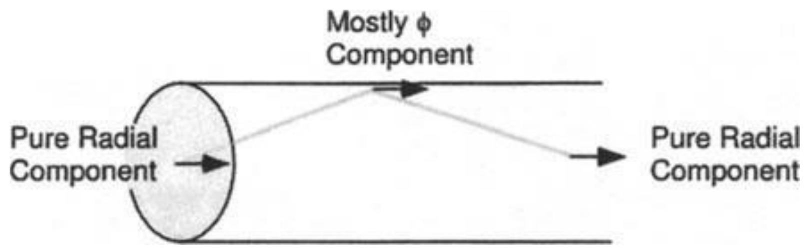
$$\vec{E}(r,\phi,z) = \hat{r}E_r(r,\phi,z) + \hat{\phi}E_\phi(r,\phi,z) + \hat{z}E_z(r,\phi,z)$$

1.3 WAVE EQUATION IN CYLINDRICAL COORDINATES

Unlike the vector Laplacian in rectilinear coordinates, ∇^2 in cylindrical coordinates can not be easily decomposed into three individual components.

This because the transverse components $\hat{r}$ e $\hat{\phi}$ of the field are tightly coupled.

Imagine for example a linearly-polarized field travelling at a slight angle to the axis of a cylindrical waveguide, as shown in the picture



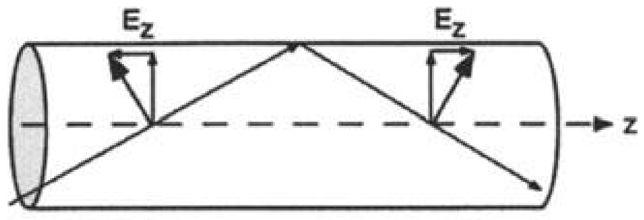
At $z = 0$, the field is purely radial, but as it travels down the axis, it becomes an azimuthal ($\hat{\phi}$) field.

It is impossible to decouple E_r e E_ϕ components.

Here is the critical point in understanding the analysis of a two-dimensional waveguide: only the $\hat{z}$ -component of a field, E_z, does not couple to other components as it propagates.

Even after reflection at a cylindrical surface, the E_z component remains oriented along the $\hat{z}$ -axis. .

1.3 WAVE EQUATION IN CYLINDRICAL COORDINATES



The longitudinal component of the electric field does not change through either propagation or reflection at the cylindrical surface.

Hence, we will attempt to find a solution for E_z using the wave equation.

Once we have a solution for $E_z(r, \phi, z)$,we can use Maxwell's equations to relate E_z to $E_r(r, \phi, z)$ and $E_\phi(r, \phi, z)$

- $E_z(r, \phi, z)$ and $H_z(r, \phi, z)$ are longitudinal fields,
- $E_r(r, \phi, z)$ and $E_\phi(r, \phi, z)$, $H_r(r, \phi, z)$ and $H_\phi(r, \phi, z)$ are transverse fields

Since E_z couples only to itself, it is possible to write the scalar wave equation for E_z directly in cylindrical coordinates: $\nabla^2 \vec{E}(\vec{r}) + k_o^2 n^2 \vec{E}(\vec{r}) = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + k_0^2 n^2 E_z = 0$$

1.4 SOLUTION OF THE WAVE EQUATION FOR E_z

Since E_z is a function of r, ϕ e z, we can employ separation of variables to solve the scalar equation. Thus we can set $E_z(r,\phi,z)$ as:

$$E_z(r,\phi,z) = R(r)\Phi(\phi)Z(z)$$

Substituting this in the Helmhotz equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + k_0^2 n^2 E_z = 0$$

$$R''\Phi\mathrm{Z} + \frac{1}{r}R'\Phi\mathrm{Z} + \frac{1}{r^2}R\Phi''\mathrm{Z} + R\Phi\mathrm{Z}'' + k_o^2 n^2 R\Phi\mathrm{Z} = 0$$

Multiply both sides by: $\frac{r^2}{R\Phi\mathrm{Z}}$:

$$r^2\frac{R''}{R} + \mathrm{r}\frac{R'}{R} + \frac{\Phi''}{\Phi} + r^2\frac{\mathrm{Z}''}{Z} + k_o^2 n^2 r^2 = 0$$

The term $\frac{\mathrm{Z}''}{Z}$ depends neither on ϕ nor on r, thus we can assume that it is proportional to a constant value:

1.4 SOLUTION OF THE WAVE EQUATION FOR E_z

$$\frac{Z''}{Z} = -\beta^2$$

The solution of this differential equation are plane waves:

$$Z(z) = Be^{-i\beta z}$$

where β is the $\hat{z}$ - component of the wavevector k in the waveguide.

We can substitute $\frac{Z''}{z} = -\beta^2$ into the wave equation:

$$r^2\frac{R''}{R} + \mathrm{r}\frac{R'}{R} + \frac{\Phi''}{\Phi} + r^2\frac{Z''}{Z} + k_o^2 n^2 r^2 = 0$$

$$r^2\frac{R''}{R} + \mathrm{r}\frac{R'}{R} + \frac{\Phi''}{\Phi} - r^2\beta^2 + k_o^2 n^2 r^2 = 0$$

We can use the same procedure also for the term $\frac{\Phi''}{\Phi}$:

$$r^2\frac{R''}{R} + \mathrm{r}\frac{R'}{R} - r^2\beta^2 + k_o^2 n^2 r^2 = -\frac{\Phi''}{\Phi} = \upsilon^2$$

where υ^2 has a constant value.

1.4 SOLUTION OF THE WAVE EQUATION FOR E_z

The equation $-\frac{\Phi''}{\Phi} = \upsilon^2$ can be solved directly for $\Phi(\phi)$:

$$\Phi(\phi) = Ae^{i\upsilon\phi}$$

where A is a normalization constant.

Since circular symmetry requires: $\Phi(\phi) = \Phi(\phi + 2\pi)$, υ must be an integer.

Substituting $\Phi(\phi) = Ae^{i\upsilon\phi}$ into the wave equation yields an equation that only contains $R(r)$:

$$r^2\frac{R''}{R} + \mathrm{r}\frac{R'}{R} - r^2\beta^2 + k_o^2n^2r^2 = \upsilon^2$$

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + r^2\left(k_0^2n^2 - \beta^2 - \frac{\upsilon^2}{r^2}\right)R = 0$$

The solutions to this differential equation is given by Bessel functions.

- When the argument $k_0^2n^2 - \beta^2 - \frac{\upsilon^2}{r^2} > 0$, solutions are **Bessel Functions of the First Kind of Order υ**, usually simbolized as $J_\nu(\kappa r)$, where $\kappa^2 = k_o^2n^2 - \beta^2$

1.4 SOLUTION OF THE WAVE EQUATION FOR E_z

$$r^2\frac{\partial^2 R}{\partial r^2}+r\frac{\partial R}{\partial r}+r^2\left(k_0^2n^2-\beta^2-\frac{\upsilon^2}{r^2}\right)R=0$$

- When $k_0^2n^2-\beta^2-\frac{\upsilon^2}{r^2}<0$, solutions are **Modified Bessel Functions of the Second Kind of Order $\boldsymbol{\upsilon}$**, usually simbolized as $K_\nu(\gamma r)$ where $\gamma^2=\beta^2-k_o^2n^2$

Plots of both types of Bessel function are shown in the following picture.

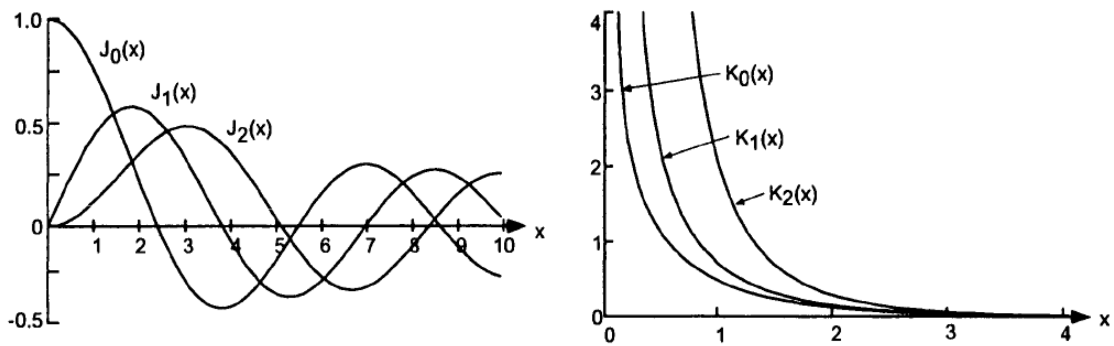


Figure 4.4. The first three Bessel functions of the first kind, $J_\nu(\kappa r)$, and of the second kind, $K_\nu(\gamma r)$.

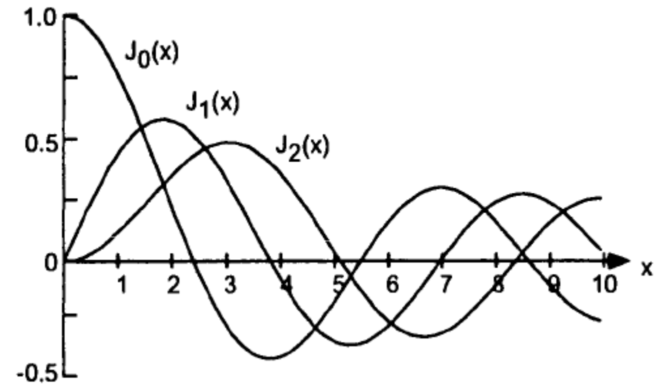
Now, let us comment the trend of both kinds of Bessel functions.

1.4 SOLUTION OF THE WAVE EQUATION FOR E_z

The $J_\nu(\kappa r)$ functions are periodic along the radial axis.

Only $J_0(\kappa r)$ has finite value at a $r = 0$, all others $J_{\nu\neq 0}(\kappa r)$ functions are zero at the origin.

For large arguments κr, the Bessel function of the first kind can be approximated as:



$$J_\nu(\kappa r) \approx \sqrt{\frac{2}{\pi \kappa r}} cos\left(\kappa r - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

These Bessel functions can be viewed as damped sine waves.

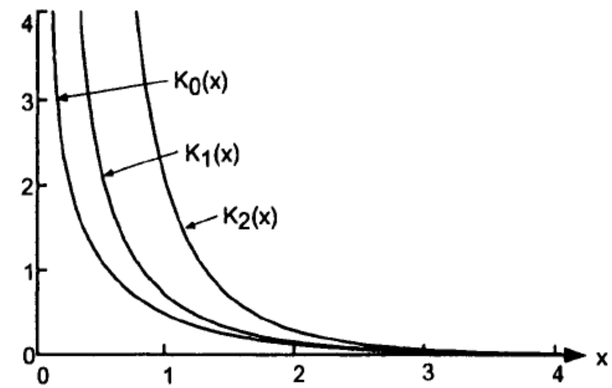
The amplitude decreases slowly with radial distance, much like the amplitude of a spreading wave in a pond.

As we shall see, the $J_\nu(\kappa r)$ Bessel functions describe the radial standing wave in a cylindrical structure.

1.4 SOLUTION OF THE WAVE EQUATION FOR Ez

The modified Bessel functions $K_\nu(\gamma r)$ display a monotonic decreasing characteristic.

The higher orders of the function decrease at a slower rate, but all orders have the same functional form.



In the limit of large γr, the functions can be approximated as:

$$K_\nu(\gamma r) \approx \frac{e^{-\gamma r}}{\sqrt{2\pi\,\gamma r}}$$

Again, this looks like a radially damped exponentially decreasing function.

Note that at large distance, all orders of $K_\nu(\gamma r)$ look approximately the same.

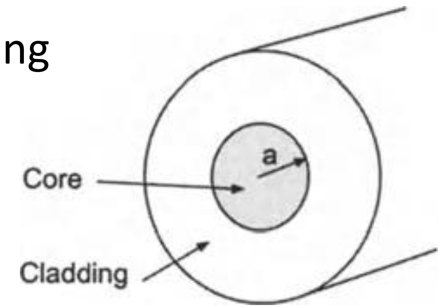
The $\frac{1}{\sqrt{2\pi\,\gamma r}}$ dependence is the natural decrease of a wave as it expands with radius, while the exponent represents decay due to evanescent interference.

$K_\nu(\gamma r)$ functions are used to describe evanescent fields in the optical waveguide.

1.5 FIELD DISTRIBUTION IN THE STEP-INDEX FIBER

In this and the next section, we derive expressions for the fields and the characteristic equation for the cylindrical dielectric waveguide.

Consider a waveguide with a core of radius a surrounded by a cladding with lower index.



Since we expect oscillatory solutions to the transverse wave equation in the core, $J_\nu(\kappa r)$ solutions will be sought in this region.

From the analysis above we can see that an oscillatory solution only occurs when β satisfies:

$$k_0 n_{core} > \beta > k_0 n_{clad}$$

$$k_0^2 n^2 - \beta^2 - \frac{v^2}{r^2} > 0$$

Outside the higher index core, the field exponentially decays, so we choose the $K_\nu(\gamma r)$ solutions for $r > a$.

The only criteria on the size of the cladding is that the evanescent field should decay to negligible values long before the outer radius of the cladding is reached.

1.5 FIELD DISTRIBUTION IN THE STEP-INDEX FIBER

Let's construct a solution to the wave equation.

The complete **longitudinal fields** $E_z(r,\phi,z)$ e $H_z(r,\phi,z)$ in both regions can be written as:

$$E_z(r,\phi,z) = \begin{cases} AJ_\nu(\kappa r)e^{i\upsilon\phi}e^{-i\beta z} + c.c. & r < a \\ CK_\nu(\gamma r)e^{i\upsilon\phi}e^{-i\beta z} + c.c. & r > a \end{cases}$$

$$H_z(r,\phi,z) = \begin{cases} BJ_\nu(\kappa r)e^{i\upsilon\phi}e^{-i\beta z} + c.c. & r < a \\ DK_\nu(\gamma r)e^{i\upsilon\phi}e^{-i\beta z} + c.c. & r > a \end{cases}$$

Note that the electric and magnetic fields have the same spatial dependence.

Also note that υ is a mode number, or eigenvalue.

Determining the coefficients A, B, C and D equires application of the boundary conditions, specifically, continuity of the tangential $\vec{E}$ and $\vec{H}$ fields.

These steps involve a lot of mathematics but are necessary for finding the eigenvalue equation of the step index fiber.

1.5 FIELD DISTRIBUTION IN THE STEP-INDEX FIBER

Boundary conditions for **longitudinal fields** $E_z(r,\phi,z)$ e $H_z(r,\phi,z)$ at $r = a$ are:

$$A J_\nu(\kappa a) e^{i\nu\phi} e^{-i\beta z} = C K_\nu(\gamma a) e^{i\nu\phi} e^{-i\beta z}$$

$$B J_\nu(\kappa a) e^{i\nu\phi} e^{-i\beta z} = D K_\nu(\gamma a) e^{i\nu\phi} e^{-i\beta z}$$

$$E_z(r,\phi,z) = \begin{cases} A J_\nu(\kappa r) e^{i\nu\phi} e^{-i\beta z} + c.c. & r < a \\ C K_\nu(\gamma r) e^{i\nu\phi} e^{-i\beta z} + c.c. & r > a \end{cases}$$

$$H_z(r,\phi,z) = \begin{cases} B J_\nu(\kappa r) e^{i\nu\phi} e^{-i\beta z} + c.c. & r < a \\ D K_\nu(\gamma r) e^{i\nu\phi} e^{-i\beta z} + c.c. & r > a \end{cases}$$

We need two more equations in order to have a resolvable equation systems composed by 4 equations with the 4 coefficients A, B, C e D .

Thus, together with boundary conditions for longitudinal field, we need to impose boundary conditions also for azimuthal field components E_ϕ e H_ϕ.

To do that, we need to find an expression for both azimuthal fields.

In the next slide, the procedure will be described.

For a complete description (with calculations), refer to the book «Allen H. Cherin, An Introduction to Optical Fibers, McGraw-Hill Book Co., New York (1983) «

1.5 FIELD DISTRIBUTION IN THE STEP-INDEX FIBER

1. From Maxwell's equations:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} = -\mu i \omega \vec{H} = -\mu i \omega \left[\hat{r} H_r + \hat{\phi} H_\phi + \hat{z} H_z \right]$$

2. Expand $\vec{\nabla} \times \vec{E}$ term in cylindrical components :

$$\vec{\nabla} \times \vec{E} = \left[\frac{1}{r} \frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} \right] \hat{r} + \left[\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \right] \hat{\phi} + \left[\frac{\partial (r E_\phi)}{\partial r} - \frac{\partial E_r}{\partial \phi} \right] \hat{z}$$

3. After collecting terms, the field components H_r, E_r, E_ϕ and H_ϕ can be described in terms of the longitudinal components E_z e H_z

$$E_\phi = \frac{-i}{\alpha^2} \left(\frac{\beta}{r} \frac{\partial E_z}{\partial \phi} - \mu\omega \frac{\partial H_z}{\partial r} \right) \qquad E_r = \frac{-i}{\alpha^2} \left(\frac{\mu\omega}{r} \frac{\partial H_z}{\partial \phi} + \beta \frac{\partial E_z}{\partial r} \right)$$

$$H_\phi = \frac{-i}{\alpha^2} \left(\omega\epsilon \frac{\partial E_z}{\partial r} + \frac{\beta}{r} \frac{\partial H_z}{\partial \phi} \right) \qquad H_r = \frac{-i}{\alpha^2} \left(\beta \frac{\partial H_z}{\partial r} - \frac{\epsilon\omega}{r} \frac{\partial E_z}{\partial \phi} \right)$$

where $\alpha^2 = k_0^2 n^2 - \beta^2$

Note that α^2 is a positive quantity in the core, and a negative quantity in the cladding for allowed values of β.

1.5 FIELD DISTRIBUTION IN THE STEP-INDEX FIBER

Longitudinal components

$$\boldsymbol{E_z}(r,\phi,z) = \begin{cases} AJ_\nu(\kappa r)e^{i\nu\phi}e^{-i\beta z} + c.c. & r < a \\ CK_\nu(\gamma r)e^{i\nu\phi}e^{-i\beta z} + c.c. & r > a \end{cases}$$

$$\boldsymbol{H_z}(r,\phi,z) = \begin{cases} BJ_\nu(\kappa r)e^{i\nu\phi}e^{-i\beta z} + c.c. & r < a \\ DK_\nu(\gamma r)e^{i\nu\phi}e^{-i\beta z} + c.c. & r > a \end{cases}$$

Radial and azimuthal components as a function of the **longitudinal components**

$$\boldsymbol{E_\phi} = \frac{-i}{\alpha^2}\left(\frac{\beta}{r}\frac{\partial \boldsymbol{E_z}}{\partial \phi} - \mu\omega\frac{\partial \boldsymbol{H_z}}{\partial r}\right)$$

$$\boldsymbol{H_\phi} = \frac{-i}{\alpha^2}\left(\omega\epsilon\frac{\partial \boldsymbol{E_z}}{\partial r} + \frac{\beta}{r}\frac{\partial \boldsymbol{H_z}}{\partial \phi}\right)$$

$$\boldsymbol{E_r} = \frac{-i}{\alpha^2}\left(\frac{\mu\omega}{r}\frac{\partial \boldsymbol{H_z}}{\partial \phi} + \beta\frac{\partial \boldsymbol{E_z}}{\partial r}\right)$$

$$\boldsymbol{H_r} = \frac{-i}{\alpha^2}\left(\beta\frac{\partial \boldsymbol{H_z}}{\partial r} - \frac{\epsilon\omega}{r}\frac{\partial \boldsymbol{E_z}}{\partial \phi}\right)$$

Inserting the longitudinal fields E_z e H_z into the expressions for H_r, E_r, E_ϕ and H_ϕ, the radial and azimuthal fields can be exactly calculated.

1.5 FIELD DISTRIBUTION IN THE STEP-INDEX FIBER

In the core region: $r < a$

$$\boldsymbol{E_r} = \frac{-i\beta}{\kappa^2}\left[A\kappa J'_\nu(\kappa r) + \frac{i\omega\mu\upsilon}{\beta r}BJ_\nu(\kappa r)\right]e^{i\upsilon\phi}e^{-i\beta z}$$

$$\boldsymbol{E_\phi} = \frac{-i\beta}{\kappa^2}\left[\frac{i\upsilon}{r}AJ_\nu(\kappa r) - \frac{\omega\mu}{\beta}B\kappa J'_\nu(\kappa r)\right]e^{i\upsilon\phi}e^{-i\beta z}$$

$$\boldsymbol{H_r} = \frac{-i\beta}{\kappa^2}\left[B\kappa J'_\nu(\kappa r) - \frac{i\omega\epsilon_{core}\upsilon}{\beta r}AJ_\nu(\kappa r)\right]e^{i\upsilon\phi}e^{-i\beta z}$$

$$\boldsymbol{H_\phi} = \frac{-i\beta}{\kappa^2}\left[\frac{i\upsilon}{r}BJ_\nu(\kappa r) + \frac{\omega\epsilon_{core}}{\beta}A\kappa J'_\nu(\kappa r)\right]e^{i\upsilon\phi}e^{-i\beta z}$$

with

$$J'_\nu(\kappa r) = \frac{dJ_\nu(\kappa r)}{d\kappa r}$$

In the cladding region: $r > a$

$$\boldsymbol{E_r} = \frac{i\beta}{\gamma^2}\left[C\gamma K'_\nu(\gamma r) + \frac{i\omega\mu\upsilon}{\beta r}DK_\nu(\gamma r)\right]e^{i\upsilon\phi}e^{-i\beta z}$$

$$\boldsymbol{E_\phi} = \frac{i\beta}{\gamma^2}\left[\frac{i\upsilon}{r}CK_\nu(\gamma r) - \frac{\omega\mu}{\beta}D\gamma K'_\nu(\gamma r)\right]e^{i\upsilon\phi}e^{-i\beta z}$$

$$\boldsymbol{H_r} = \frac{i\beta}{\gamma^2}\left[D\gamma K'_\nu(\gamma r) - \frac{i\omega\epsilon_{clad}\upsilon}{\beta r}CK_\nu(\gamma r)\right]e^{i\upsilon\phi}e^{-i\beta z}$$

$$\boldsymbol{H_\phi} = \frac{i\beta}{\gamma^2}\left[\frac{i\upsilon}{r}DK_\nu(\gamma r) + \frac{\omega\epsilon_{clad}}{\beta}C\gamma K'_\nu(\gamma r)\right]e^{i\upsilon\phi}e^{-i\beta z}$$

with

$$K'_\nu(\kappa r) = \frac{dK_\nu(\kappa r)}{d\kappa r}$$

1.6 BOUNDARY CONDITIONS FOR THE STEP-INDEX WAVEGUIDE

To determine β, and the coefficients, A, B, C, and D, we need to apply the boundary conditions.

$$A J_\nu(\kappa a) e^{i\nu\phi} e^{-i\beta z} = C K_\nu(\gamma a) e^{i\nu\phi} e^{-i\beta z}$$

$$B J_\nu(\kappa a) e^{i\nu\phi} e^{-i\beta z} = D K_\nu(\gamma a) e^{i\nu\phi} e^{-i\beta z}$$

The boundary conditions at $r = a$ require that the four tangential components, H_r, E_r, E_ϕ and H_ϕ be continuous at the core-cladding boundary.

The boundary condition for E_ϕ is:

$$\boldsymbol{E_\phi} = \begin{cases} \dfrac{-i\beta}{\kappa^2}\left[\dfrac{i\nu}{r} A J_\nu(\kappa r) - \dfrac{\omega\mu}{\beta} B\kappa J'_\nu(\kappa r)\right] e^{i\nu\phi} e^{-i\beta z} & r < a \\ \dfrac{i\beta}{\gamma^2}\left[\dfrac{i\nu}{r} C K_\nu(\gamma r) - \dfrac{\omega\mu}{\beta} D\gamma K'_\nu(\gamma r)\right] e^{i\nu\phi} e^{-i\beta z} & r > a \end{cases}$$

$$\frac{\beta\nu}{\kappa^2 a} A J_\nu(\kappa a) + \frac{i\omega\mu}{\kappa} B J'_\nu(\kappa a) = -\frac{\beta\nu}{\gamma^2 a} C K_\nu(\gamma a) - \frac{i\omega\mu}{\gamma} D K'_\nu(\gamma a)$$

and for H_ϕ :

$$\boldsymbol{H_\phi} = \begin{cases} \dfrac{-i\beta}{\kappa^2}\left[\dfrac{i\nu}{r} B J_\nu(\kappa r) + \dfrac{\omega\epsilon_{core}}{\beta} A\kappa J'_\nu(\kappa r)\right] e^{i\nu\phi} e^{-i\beta z} & r < a \\ \dfrac{i\beta}{\gamma^2}\left[\dfrac{i\nu}{r} D K_\nu(\gamma r) + \dfrac{\omega\epsilon_{clad}}{\beta} C\gamma K'_\nu(\gamma r)\right] e^{i\nu\phi} e^{-i\beta z} & r > a \end{cases}$$

$$-i\frac{\omega\epsilon_{core}}{\kappa} A J'_\nu(\kappa a) + \frac{\beta\nu}{\kappa^2 a} B J_\nu(\kappa a) = \frac{i\omega\epsilon_{clad}}{\gamma} C K'_\nu(\gamma a) - \frac{\nu\beta}{\gamma^2 a} D K_\nu(\gamma a)$$

1.6 BOUNDARY CONDITIONS FOR THE STEP-INDEX WAVEGUIDE

Now we have an equation systems composed by 4 equations with the 4 coefficients A, B, C and D:

$$AJ_\nu(\kappa a)e^{i\upsilon\phi}e^{-i\beta z} = CK_\nu(\gamma a)e^{i\upsilon\phi}e^{-i\beta z}$$

$$BJ_\nu(\kappa a)e^{i\upsilon\phi}e^{-i\beta z} = DK_\nu(\gamma a)e^{i\upsilon\phi}e^{-i\beta z}$$

$$\frac{\beta\upsilon}{\kappa^2 a}AJ_\nu(\kappa a) + \frac{i\omega\mu}{\kappa}BJ'_\nu(\kappa a) = -\frac{\beta\upsilon}{\gamma^2 a}CK_\nu(\gamma a) - \frac{i\omega\mu}{\gamma}DK'_\nu(\gamma a)$$

$$-i\frac{\omega\epsilon_{core}}{\kappa}AJ'_\nu(\kappa a) + \frac{\beta\upsilon}{\kappa^2 a}BJ_\nu(\kappa a) = \frac{i\omega\epsilon_{clad}}{\gamma}CK'_\nu(\gamma a) - \frac{\upsilon\beta}{\gamma^2 a}DK_\nu(\gamma a)$$

The simplest way to simultaneously satisfy all four boundary value equations is to write the four linear equations in matrix form, and then set the determinant of the matrix equal to zero.

$$\begin{pmatrix} J_\nu(\kappa a) & 0 & -K_\nu(\gamma a) & 0 \\ 0 & J_\nu(\kappa a) & 0 & -K_\nu(\gamma a) \\ \frac{\beta\upsilon}{\kappa^2 a}J_\nu(\kappa a) & \frac{i\omega\mu}{\kappa}J'_\nu(\kappa a) & \frac{\beta\upsilon}{\gamma^2 a}K_\nu(\gamma a) & \frac{i\omega\mu}{\gamma}K'_\nu(\gamma a) \\ -i\frac{\omega\epsilon_{core}}{\kappa}J'_\nu(\kappa a) & \frac{\beta\upsilon}{\kappa^2 a}J_\nu(\kappa a) & -\frac{i\omega\epsilon_{clad}}{\gamma}K'_\nu(\gamma a) & \frac{\upsilon\beta}{\gamma^2 a}K_\nu(\gamma a) \end{pmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$$

1.6 BOUNDARY CONDITIONS FOR THE STEP-INDEX WAVEGUIDE

For non-trivial solutions (i.e., non-zero amplitudes), the four equations will simultaneously equal zero if and only if the determinant of the matrix equals zero.

Expansion of the determinant yields the "**characteristic equation**" for the step-index fiber.

$$\frac{\beta^2 v^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2=\left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\cdot\left[\frac{k_o^2 n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_o^2 n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

This formidable equation requires numerical or graphical solution.

There is only one unknown: β, because κ and γ are functions of β and the local index.

$$\kappa^2 = k_o^2 n_{core}^2 - \beta^2$$

$$\gamma^2 = \beta^2 - k_o^2 n_{clad}^2$$

Due to the oscillatory nature of $J_\nu(\kappa a)$, there can be several values of β for a given structure.

Since there are two dimensional degrees of freedom in the cylindrical waveguide, solutions to the wave equation are labelled with two indices, v and m. Both numbers are integers.

1.6 BOUNDARY CONDITIONS FOR THE STEP-INDEX WAVEGUIDE

$$\boldsymbol{E_z}(r,\phi,z) = \begin{cases} AJ_\nu(\kappa r)e^{i\boldsymbol{\nu}\phi}e^{-i\boldsymbol{\beta_m}z} + c.c. & r < a \\ CK_\nu(\gamma r)e^{i\boldsymbol{\nu}\phi}e^{-i\boldsymbol{\beta_m}z} + c.c. & r > a \end{cases}$$

The $\boldsymbol{m}$ value is called the ***radial mode number*** and represents the number of radial nodes that exist in the field distribution.

The integer $\boldsymbol{\nu}$ is called the ***angular mode number*** *and* represents the number of angular nodes that exist in the field distribution.

Once β is determined from the characteristic equation, three of the coefficients (A, B, C and D) can be determined in terms of the fourth by solving the individual equations.

$$\frac{\beta^2\upsilon^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2 = \left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\cdot\left[\frac{k_o^2 n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_o^2 n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

From the boundary conditions for continuity of E_z at $r = a$

$$AJ_\nu(\kappa a) = CK_\nu(\gamma a)$$

$$BJ_\nu(\kappa a) = DK_\nu(\gamma a)$$

$$AJ_\nu(\kappa a)e^{i\nu\phi}e^{-i\beta z} = CK_\nu(\gamma a)e^{i\nu\phi}e^{-i\beta z}$$

$$BJ_\nu(\kappa a)e^{i\nu\phi}e^{-i\beta z} = DK_\nu(\gamma a)e^{i\nu\phi}e^{-i\beta z}$$

1.6 BOUNDARY CONDITIONS FOR THE STEP-INDEX WAVEGUIDE

where

$$C = \frac{J_\nu(\kappa a)}{K_\nu(\gamma a)} A$$

$$D = \frac{J_\nu(\kappa a)}{K_\nu(\gamma a)} B$$

$$AJ_\nu(\kappa a) = CK_\nu(\gamma a)$$

$$BJ_\nu(\kappa a) = DK_\nu(\gamma a)$$

The coefficients A and B can be related to one another using the continuity of E_ϕ

$$\frac{\beta \upsilon}{\kappa^2 a} AJ_\nu(\kappa a) + \frac{i\omega\mu}{\kappa} BJ'_\nu(\kappa a) = -\frac{\beta \upsilon}{\gamma^2 a} CK_\nu(\gamma a) - \frac{i\omega\mu}{\gamma} DK'_\nu(\gamma a)$$

By using the expressions above for C and D, the continuity of E_ϕ becomes:

$$\frac{\beta \upsilon}{\kappa^2 a} AJ_\nu(\kappa a) + \frac{i\omega\mu}{\kappa} BJ'_\nu(\kappa a) = -\frac{\beta \upsilon}{\gamma^2 a} K_\nu(\gamma a) \frac{J_\nu(\kappa a)}{K_\nu(\gamma a)} A - \frac{i\omega\mu}{\gamma} \frac{J_\nu(\kappa a)}{K_\nu(\gamma a)} BK'_\nu(\gamma a)$$

which leads to:

$$A\left[\frac{\beta \upsilon}{\kappa^2 a} J_\nu(\kappa a) + \frac{\beta \upsilon}{\gamma^2 a} J_\nu(\kappa a)\right] = B\left[-\frac{i\omega\mu}{\kappa} J'_\nu(\kappa a) - \frac{i\omega\mu}{\gamma} \frac{J_\nu(\kappa a)}{K_\nu(\gamma a)} K'_\nu(\gamma a)\right]$$

1.6 BOUNDARY CONDITIONS FOR THE STEP-INDEX WAVEGUIDE

$$A\left[\frac{\beta\upsilon}{\kappa^2 a}J_\nu(\kappa a)+\frac{\beta\upsilon}{\gamma^2 a}J_\nu(\kappa a)\right]=B\left[-\frac{i\omega\mu}{\kappa}J'_\nu(\kappa a)-\frac{i\omega\mu}{\gamma}\frac{J_\nu(\kappa a)}{K_\nu(\gamma a)}K'_\nu(\gamma a)\right]$$

This can be arranged as:

$$B=\frac{\frac{i\beta\upsilon}{a\omega\mu}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]}{\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}}A$$

Coversely, if we use the continuity of H_ϕ to relate coefficients A and B to one another:

$$-i\frac{\omega\epsilon_{core}}{\kappa}AJ'_\nu(\kappa a)+\frac{\beta\upsilon}{\kappa^2 a}BJ_\nu(\kappa a)=\frac{i\omega\epsilon_{clad}}{\gamma}CK'_\nu(\gamma a)-\frac{\upsilon\beta}{\gamma^2 a}DK_\nu(\gamma a)$$

with same steps as before, one has:

$$B=\frac{\frac{i\omega a}{\beta\upsilon}\left[\frac{n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]}{\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]}A$$

1.6 BOUNDARY CONDITIONS FOR THE STEP-INDEX WAVEGUIDE

By the continuity of E_ϕ

$$B = \frac{\frac{i\beta\upsilon}{a\omega\mu}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]}{\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}}A$$

By the continuity of H_ϕ

$$B = \frac{\frac{i\omega a}{\beta\upsilon}\left[\frac{n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]}{\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]}A$$

The choice of which equation to use depends on the type of mode carried in the waveguide.

Note that B/A is purely imaginary in both cases, indicating that the two longitudinal fields are $\pi/2$ out of phase.

The quantity B/A is of particular interest in determining the relative size of the longitudinal components of the $\vec{E}$ and $\vec{H}$ fields.

These, in tum, characterize the type of mode.

1.7 TRANSVERSE ELECTRIC AND TRANSVERSE MAGNETIC MODES

Consider the characteristic equation

$$\frac{\beta^2 \upsilon^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2 = \left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\cdot\left[\frac{k_o^2 n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_o^2 n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

For the case where $\upsilon = 0$.

$$\left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\cdot\left[\frac{k_o^2 n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_o^2 n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right] = 0$$

Either term on the left-hand side can be set to zero to satisfy the equation.

The two terms in square bracket appear individually in expressions where amplitude A was related to amplitude B.

$$B = \frac{\frac{i\beta\upsilon}{a\omega\mu}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]}{\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}} A$$

$$B = \frac{\frac{i\omega a}{\beta\upsilon}\left[\frac{n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]}{\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]} A$$

1.7 TRANSVERSE ELECTRIC AND TRANSVERSE MAGNETIC MODES

$$\left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\cdot\left[\frac{k_o^2 n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_o^2 n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]=0$$

$$B=\frac{\frac{i\beta\upsilon}{a\omega\mu}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]}{\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}}A$$

$$B=\frac{\frac{i\omega a}{\beta\upsilon}\left[\frac{n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]}{\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]}A$$

- If $\left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]=0$, then A must also be zero to keep the magnitude of B in finite. With $A=0$, $E_z=0$ and the electric field will be transverse. Such modes are called **TE modes**.

$$\boldsymbol{E_z}(r,\phi,z)=\begin{cases}AJ_\nu(\kappa r)e^{i\upsilon\phi}e^{-i\beta z}+c.c. & r<a\\ CK_\nu(\gamma r)e^{i\upsilon\phi}e^{-i\beta z}+c.c. & r>a\end{cases}$$

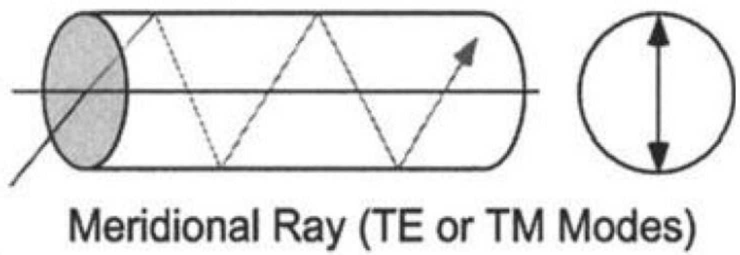
$$\boldsymbol{H_z}(r,\phi,z)=\begin{cases}BJ_\nu(\kappa r)e^{i\upsilon\phi}e^{-i\beta z}+c.c. & r<a\\ DK_\nu(\gamma r)e^{i\upsilon\phi}e^{-i\beta z}+c.c. & r>a\end{cases}$$

- If $\left[\frac{k_o^2 n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_o^2 n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]=0$, then the magnitude of B will be zero and the longitudinal component of the $\vec{H}$ field will be zero. Such modes are called **TM modes**.

Thus, if $\upsilon\ =\ 0$, the allowed modes will be either **TE** or **TM**.

1.7 TRANSVERSE ELECTRIC AND TRANSVERSE MAGNETIC MODES

How **TE** or **TM** modes appear?



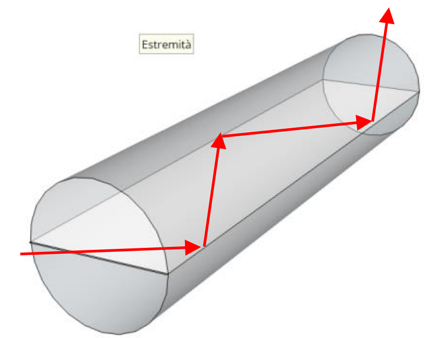


Figure 4.7. A meridional ray zig-zags down the fiber, passing through the origin. There is no angular rotation of the ray path as it propagates.

All multiple reflections at core-cladding interface will lie in one plane and the electric field will be orthogonal to that plane.

The problem of finding the allowed values of the propagation vector β reduces to finding the roots of the characteristic equation:

$$\left[\frac{J'_{\nu}(\kappa a)}{\kappa J_{\nu}(\kappa a)} + \frac{K'_{\nu}(\gamma a)}{\gamma K_{\nu}(\gamma a)}\right] \cdot \left[\frac{k_o^2 n_{core}^2 J'_{\nu}(\kappa a)}{\kappa J_{\nu}(\kappa a)} + \frac{k_o^2 n_{clad}^2 K'_{\nu}(\gamma a)}{\gamma K_{\nu}(\gamma a)}\right] = 0$$

1.7 TRANSVERSE ELECTRIC AND TRANSVERSE MAGNETIC MODES

$$\left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\cdot\left[\frac{k_o^2 n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_o^2 n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]=0$$

These equations for the TE and TM modes can be further simplified using the Bessel function relations.

$$\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}=\pm\frac{J_{\nu\mp 1}(\kappa a)}{\kappa J_\nu(\kappa a)}\mp\frac{\nu}{\kappa^2}$$

$$\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}=\mp\frac{K_{\nu\pm 1}(\kappa a)}{\gamma K_\nu(\gamma a)}\mp\frac{\nu}{\gamma^2}$$

Consider first the ***TE* mode**.

The first term of characteristic equation should be set equal to zero.

Using the first of two Bessel function relations, **the eigenvalue equation for TE modes** becomes ($\upsilon=0$):

$$-\frac{J_1(\kappa a)}{\kappa J_0(\kappa a)}-\frac{K_1(\kappa a)}{\gamma K_0(\gamma a)}=0$$

1.7 TRANSVERSE ELECTRIC AND TRANSVERSE MAGNETIC MODES

$$\left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right] \cdot \left[\frac{\boldsymbol{k_o^2 n_{core}^2 J'_\nu(\kappa a)}}{\boldsymbol{\kappa J_\nu(\kappa a)}} + \frac{\boldsymbol{k_o^2 n_{clad}^2 K'_\nu(\gamma a)}}{\boldsymbol{\gamma K_\nu(\gamma a)}}\right] = 0$$

These equations for the TE and TM modes can be further simplified using the Bessel function relations.

$$\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} = \pm \frac{J_{\nu \mp 1}(\kappa a)}{\kappa J_\nu(\kappa a)} \mp \frac{\nu}{\kappa^2}$$

$$\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)} = \mp \frac{K_{\nu \pm 1}(\kappa a)}{\gamma K_\nu(\gamma a)} \mp \frac{\nu}{\gamma^2}$$

Consider now the ***TM* mode**.

The second term of characteristic equation should be set equal to zero.

Using the second of two Bessel function relations, **the eigenvalue equation for TM modes** becomes ($\upsilon = 0$):

$$-\frac{k_o^2 n_{core}^2 J_1(\kappa a)}{\kappa J_0(\kappa a)} - \frac{k_o^2 n_{clad}^2 K_1(\kappa a)}{\gamma K_0(\gamma a)} = 0$$

EXERCISES

Exercise 1

Let's consider a step-index circular fiber with a core index $n_{core} = 1.5$, a cladding index $n_{clad} = 1.45$, and with a core radius $a = 5\ \mu m$. The wavelength of the light is $\lambda = 1.3\ \mu m$.

Determine the allowed eigenvalues for the propagation vector β of the TE modes.

Suggestions:

- Use the characteristic equation for **TE modes**:

$$-\frac{J_1(\kappa a)}{\kappa J_0(\kappa a)} - \frac{K_1(\kappa a)}{\gamma K_0(\gamma a)} = 0$$

- Use the relation between κ and β

$$\kappa^2 = k_o^2 n^2 - \beta^2$$

1.8 THE HYBRID MODES

When $\upsilon \neq 0$, the characteristic equation is a little more complicated to solve.

$$\frac{\beta^2\upsilon^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2=\left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\cdot\left[\frac{k_o^2 n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_o^2 n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

The values of β will correspond to modes which have finite components of both E_z e H_z and are therefore neither TE nor TM modes.

These modes are called **EH** or **HE** modes, depending on the relative magnitude of the longitudinal $\vec{E}$ and $\vec{H}$ components

- If *A > B* the mode is called an HE mode *(E_z dominates H_z)*
- If *A < B* the mode is called an EH mode *(H_z dominates E_z)*

The EH and HE modes are called "hybrid" modes, because they have both longitudinal $\vec{E}$ and $\vec{H}$ components in the waveguide.

The EH and HE modes exist only for $\nu \geq 1$, so they have azimuthal structure.

1.8 THE HYBRID MODES

In the ray picture, these modes are called "skew" rays, because they travel down the waveguide in a screw-like pattern (Fig. 4.8), glancing off the interface as they spiral down the axis.

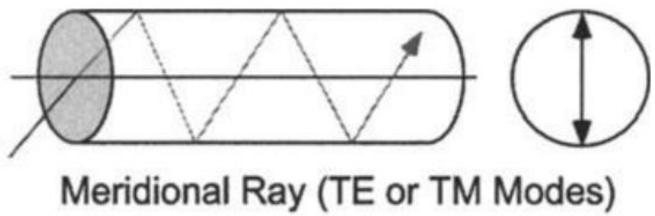


Figure 4.7. A meridional ray zig-zags down the fiber, passing through the origin. There is no angular rotation of the ray path as it propagates.

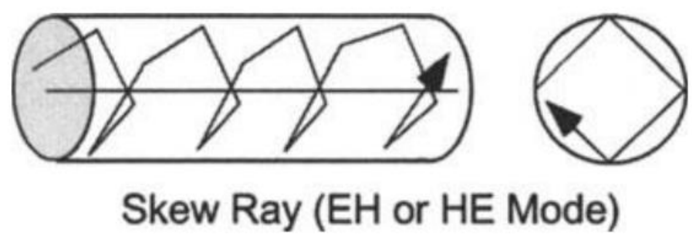


Figure 4.8. A skew ray travels in a spiral path down the fiber. The ray does not go through the origin.

The next subsection develops a useful approximation that simplifies both the calculation and visualization of the hybrid modes.

1.9 THE LINEARLY POLARIZED MODES

The characteristic equation for the hybrid modes is difficult to solve for β.

Fortunately, a very simple and reasonable approximation makes solution straightforward.

Consider again the characteristic equation:

$$\frac{\beta^2 \nu^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2 = \left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right] \cdot \left[\frac{k_o^2 n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{k_o^2 n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

For $\nu = 1, 2, \ldots$, HE and EH modes are possible.

Unfortunately, even with powerful software, finding the roots of this equation is very difficult.

Dramatic simplification occurs if we make the *weakly guiding* approximation.

For many practical optical fibers, the core and cladding index are nearly identical.

In view of this, it is not unreasonable (at least for the purpose of finding roots) to approximate that the core and cladding index are identical:

$$n_{core} \approx n_{clad} = n$$

1.9 THE LINEARLY POLARIZED MODES

$$\frac{\beta^2 \upsilon^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2 = \left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\cdot\left[\frac{k_o^2 n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_o^2 n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

In the weakly guiding approximation, the characteristic equation reduces to:

$$\frac{\beta^2 \upsilon^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2 = k_o^2 n^2\left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]^2$$

This can be further simplified noting that if $n_{core} \approx n_{clad}$, then $\beta^2 = k_o^2 n^2$, and these terms can be cancelled from both sides, leading to:

$$k_0 n_{core} > \beta > k_0 n_{clad}$$

$$\frac{\upsilon^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2 = \left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]^2$$

that can be arranged as:

$$\frac{\upsilon}{a\gamma^2}+\frac{\upsilon}{a\kappa^2} = \frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}$$

Taking advantage of some Bessel function identities

1.9 THE LINEARLY POLARIZED MODES

$$\frac{\upsilon}{a\gamma^2} + \frac{\upsilon}{a\kappa^2} = \frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}$$

Taking advantage of some Bessel function identities

$$\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} = \pm \frac{J_{\nu\mp1}(\kappa a)}{\kappa a J_\nu(\kappa a)} \mp \frac{\nu}{a\kappa^2}$$

$$\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)} = \mp \frac{K_{\nu\pm1}(\gamma a)}{a\gamma K_\nu(\gamma a)} \mp \frac{\nu}{a\gamma^2}$$

we have:

$$\frac{J_{\nu\pm1}(\kappa a)}{\kappa J_\nu(\kappa a)} = \mp \frac{K_{\nu\pm1}(\gamma a)}{\gamma K_\nu(\gamma a)}$$

These are the characteristic equations for the EH (top sign) and HE (bottom sign) modes. Solution will yield the eigenvalues for the allowed modes.

1.9 THE LINEARLY POLARIZED MODES

$$\frac{J_{\nu\pm1}(\kappa a)}{\kappa J_\nu(\kappa a)} = \mp \frac{K_{\nu\pm1}(\gamma a)}{\gamma K_\nu(\gamma a)}$$

A little more manipulation with Bessel function identities reduces these two equations into one single equation, with the index j identifies the mode:

$$\kappa \frac{J_{j-1}(\kappa a)}{J_j(\kappa a)} = -\gamma \frac{K_{j-1}(\gamma a)}{K_j(\gamma a)}$$

$j = 1$	TE, TM modes
$j = \nu + 1$	EH_υ modes
$j = \nu - 1$	HE_υ modes

Two different modes can have the same eigenvalue, namely they are degenerate.

In the weakly guiding approximation, the TE_{0m} is degenerate with the TM_{0m}, (will have the same eigenvalue β) and will propagate at the same velocity (at least to the accuracy of the weakly guiding approximation).

Similarly, the $HE_{\upsilon+1,m}$ mode and $EH_{\upsilon-1,m}$ are degenerate.

Since degenerate modes travel at the same velocity, it is possible to define stable superpositions of different modes. Certain combinations of degenerate modes can be found which are **linearly polarized** (**LP modes**).

1.9 THE LINEARLY POLARIZED MODES

Let's start by describing an idealized transverse electric field inside a step-index fiber, polarized in the $\hat{y}$-direction

$$E_y(r,\phi,z) = \hat{y}E_0 J_\nu(\kappa r)\cos(\upsilon\phi)e^{-i\beta z}$$

where E_0 is the amplitude, and the functional form is consistent with the fields we defined in for cylindrical symmetry.

$$\boldsymbol{E_z}(r,\phi,z) = \begin{cases} AJ_\nu(\kappa r)e^{i\upsilon\phi}e^{-i\beta z} + c.c. & r < a \\ CK_\nu(\gamma r)e^{i\upsilon\phi}e^{-i\beta z} + c.c. & r > a \end{cases}$$

We have assumed that the azimuthal dependence is in the form of a cosine term.

If the electric field is travelling in the $\hat{z}$ -direction and polarized in the $\hat{y}$ -direction, then the magnetic field must also be transverse and oriented in the $\hat{x}$ -direction.

Defining the impedance of the medium as:

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

we can write an expression for $H_x(r,\phi,z)$ in terms of E_y

$$H_x(r,\phi,z) = \hat{y}\frac{E_y}{\eta}J_\nu(\kappa r)cos(\upsilon\phi)e^{-i\beta z}$$

1.9 THE LINEARLY POLARIZED MODES

Since we have simply "assumed" a transverse field, it would be valuable to verify that the longitudinal component E_z is negligibly small.

The longitudinal component can be found using Faraday's equation

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

Since $\vec{D} = \epsilon \vec{E}$ and assuming that $\vec{E}$ is a time harmonic field with angular frequency ω:

$$\vec{\nabla} \times \vec{H} = -i\omega\epsilon\vec{E}$$

Expanding the curl equation in Cartesian coordinates:

$$\hat{x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - \hat{y}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = -i\omega\epsilon\left(E_x\hat{x} + E_y\hat{y} + E_z\hat{z}\right)$$

Since $\vec{H}$ has a component only in the $\hat{x}$-direction, then E_z will be:

$$H_x(r,\phi,z) = \hat{y}\frac{E_y}{\eta}J_\nu(\kappa r)cos(\upsilon\phi)e^{-i\beta z}$$

$$E_z(r,\phi,z) = \frac{1}{i\omega\epsilon}\frac{\partial H_x}{\partial y} = \frac{1}{i\omega\epsilon}\frac{\partial}{\partial y}\left[\frac{E_y}{\eta}J_\nu(\kappa r)cos(\upsilon\phi)e^{-i\beta z}\right]$$

1.9 THE LINEARLY POLARIZED MODES

$$E_z(r,\phi,z) = \frac{1}{i\omega\epsilon}\frac{\partial H_x}{\partial y} = \frac{1}{i\omega\epsilon}\frac{\partial}{\partial y}\left[\frac{E_y}{\eta}J_\nu(\kappa r)cos(\upsilon\phi)e^{-i\beta z}\right]$$

To evaluate this derivative, the operator $\frac{\partial}{\partial y}$ must be written in cylindrical coordinates:

$$\begin{cases} x = rcos\phi \\ y = rsen\phi \\ z = z \end{cases} \quad \text{or} \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \phi = arctg\left(\frac{y}{x}\right) \\ z = z \end{cases}$$

The derivative $\frac{d}{dy}$ can be written as: $\frac{d}{dy} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$

Let us calculate the two partial derivative separately:

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\left[\sqrt{x^2+y^2}\right] = \frac{1}{2\sqrt{x^2+y^2}}2y = \frac{2rsen\phi}{2r} = sen\phi$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left[arctg\left(\frac{y}{x}\right)\right] = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x^2}{xr^2} = \frac{rcos\phi}{r^2} = \frac{cos\phi}{r}$$

STEP-INDEX CIRCULAR WAVEGUIDES

1.9 THE LINEARLY POLARIZED MODES

$$\frac{d}{dy} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

$$\frac{\partial r}{\partial y} = sen\phi \qquad \frac{\partial \phi}{\partial y} = \frac{cos\phi}{r}$$

The derivative $\frac{d}{dy}$ becomes:

$$\frac{d}{dy} = sen\phi\frac{\partial}{\partial r} + \frac{cos\phi}{r}\frac{\partial}{\partial \phi}$$

Applying the derivative operator to $E_z(r,\phi,z) = \frac{1}{i\omega\epsilon}\frac{\partial}{\partial y}\left[\frac{E_y}{\eta}J_\nu(\kappa r)\cos(\upsilon\phi)e^{-i\beta z}\right]$:

$$E_z(r,\phi,z) = \frac{E_y}{i\eta\omega\epsilon}\left[sen\phi\kappa J'_\nu(\kappa r)\cos(\upsilon\phi) - \frac{\upsilon cos\phi}{r}J_\nu(\kappa r)\text{sen}(\upsilon\phi)\right]e^{-i\beta z}$$

This can be simplified with two Bessel function identities:

$$J'_\nu(\kappa r) = \frac{1}{2}[J_{\upsilon-1}(\kappa r) - J_{\upsilon+1}(\kappa r)]$$

$$\frac{\upsilon}{\kappa r}J_\nu(\kappa r) = \frac{1}{2}[J_{\upsilon-1}(\kappa r) + J_{\upsilon+1}(\kappa r)]$$

1.9 THE LINEARLY POLARIZED MODES

leading to:

$$E_z(r,\phi,z) = \frac{E_y}{i\eta\omega\epsilon}\left[\cos(\upsilon\phi)\,sen\phi\,\frac{\kappa}{2}J_{\upsilon-1}(\kappa r) - \cos(\upsilon\phi)\,sen\phi\,\frac{\kappa}{2}J_{\upsilon+1}(\kappa r) - \frac{\kappa}{2}J_{\upsilon-1}(\kappa r)cos\phi\,\mathrm{sen}(\upsilon\phi) - \frac{\kappa}{2}J_{\upsilon+1}(\kappa r)cos\phi\,\mathrm{sen}(\upsilon\phi)\right]e^{-i\beta z}$$

Collecting terms proportional to $J_{\upsilon-1}(\kappa r)$ and $J_{\upsilon+1}(\kappa r)$, separately:

$$E_z(r,\phi,z) = \frac{E_y}{i\eta\omega\epsilon}\frac{\kappa}{2}\{J_{\upsilon-1}(\kappa r)[sen\phi\,\cos(\upsilon\phi) - \mathrm{sen}(\upsilon\phi)cos\phi] - J_{\upsilon+1}(\kappa r)[sen\phi\,\cos(\upsilon\phi) + \mathrm{sen}(\upsilon\phi)cos\phi]\}e^{-i\beta z}$$

Let’s use the well-known trigonometric identity:

$$sen\alpha cos\beta = \frac{1}{2}[sen(\alpha+\beta) + sen(\alpha-\beta)]$$

for both terms in square bracket.

1.9 THE LINEARLY POLARIZED MODES

$$E_z(r,\phi,z) = \frac{E_y}{i\eta\omega\epsilon}\frac{\kappa}{2}\{J_{\upsilon-1}(\kappa r)[sen\phi\cos(\upsilon\phi) - \mathrm{sen}(\upsilon\phi)cos\phi] - J_{\upsilon+1}(\kappa r)[sen\phi\cos(\upsilon\phi) + \mathrm{sen}(\upsilon\phi)cos\phi]\}e^{-i\beta z}$$

$$sen\alpha cos\beta = \frac{1}{2}[sen(\alpha+\beta) + sen(\alpha-\beta)]$$

$$[sen\phi\cos(\upsilon\phi) - \mathrm{sen}(\upsilon\phi)cos\phi] = \frac{1}{2}[sen(\phi+\upsilon\phi) + sen(\phi-\upsilon\phi) - sen(\upsilon\phi+\phi) - sen(\upsilon\phi-\phi)]$$
$$= sen(\upsilon\phi-\phi)$$

$$[sen\phi\cos(\upsilon\phi) + \mathrm{sen}(\upsilon\phi)cos\phi] = \frac{1}{2}[sen(\phi+\upsilon\phi) + sen(\phi-\upsilon\phi) + sen(\upsilon\phi+\phi) + sen(\upsilon\phi-\phi)]$$
$$= sen(\upsilon\phi+\phi)$$

Substituting these into equation above, and cancelling terms yields:

$$E_z(r,\phi,z) = \frac{E_y}{i\eta\omega\epsilon}\frac{\kappa}{2}\{J_{\upsilon-1}(\kappa r)sen[(\upsilon-1)\phi] - J_{\upsilon+1}(\kappa r)sen[(\upsilon+1)\phi]\}e^{-i\beta z}$$

1.9 THE LINEARLY POLARIZED MODES

$$E_z(r,\phi,z) = \frac{E_y}{i\eta\omega\epsilon}\frac{\kappa}{2}\{J_{\upsilon-1}(\kappa r)sen[(\upsilon-1)\phi] - J_{\upsilon+1}(\kappa r)sen[(\upsilon+1)\phi]\}e^{-i\beta z}$$

Recall the general modal solution for the longitudinal field is described as

$$E_z(r,\phi,z) = AJ_\nu(\kappa r)cos(\upsilon\phi)e^{-i\beta z}$$

We can see by inspection that E_z is, in fact, a superposition of two modes, one with index $\upsilon+1$ nd the other with index $\upsilon-1$.

Recall that in the weakly guiding approximation, the $HE_{\upsilon+1,m}$ is degenerate with the $EH_{\upsilon-1,m}$.

This shows that it is possible to add two modes in such a way that the residual longitudinal component of the field is essentially zero.

The coefficient A describing the amplitude term can be expressed as:

$$A = E_y\frac{\kappa}{2\eta\omega\epsilon} = E_y\frac{\kappa}{2k_0 n}$$

Since the transverse wave vector κ is much smaller than the wavevector k_0 there is little amplitude in the longitudinal field.

Thus, our initial assumption of a perfectly transverse field (no longitudinal components) is nearly satisfied through proper superposition of degenerate hybrid modes.

1.9 THE LINEARLY POLARIZED MODES

From the example, we can see how two modes can be combined to create a linearly-polarized, cartesian-coordinate referenced, electric field distribution.

These superposition modes are called the $LP_{\upsilon m}$ modes.

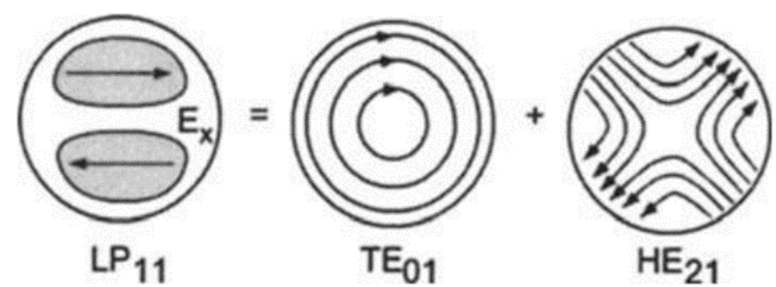
The designation and construction of an LP mode is as follows::

LP_{1m} → sum of TE_{0m}, TM_{0m} and HE_{2m} modes

$LP_{\upsilon m}$ → sum of $HE_{\upsilon+1,m}$ e $EH_{\upsilon-1,m}$ modes

LP_{0m} → HE_{1m} mode only

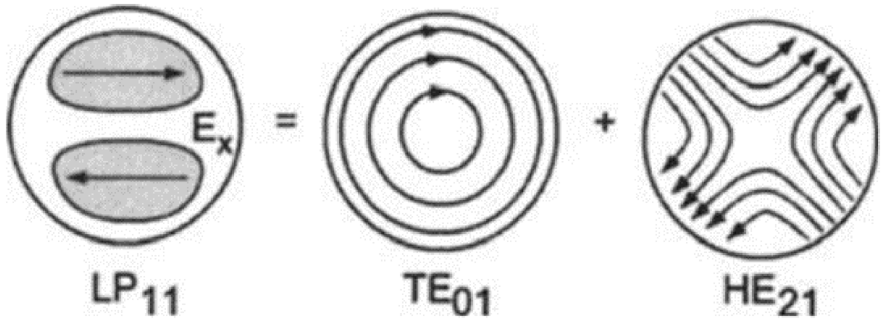
Figure shows a sketch of the mode structure of an LP_{11} mode, and a sketch of the two modes that are combined to form it.



The mode profiles of the HE_{21} and TE_{01} modes are best described in cylindrical coordinates, one having purely azimuthal fields, and the other having radial fields.

However, the superposition leads to a mode with two lobes that is linearly polarized.

1.9 THE LINEARLY POLARIZED MODES



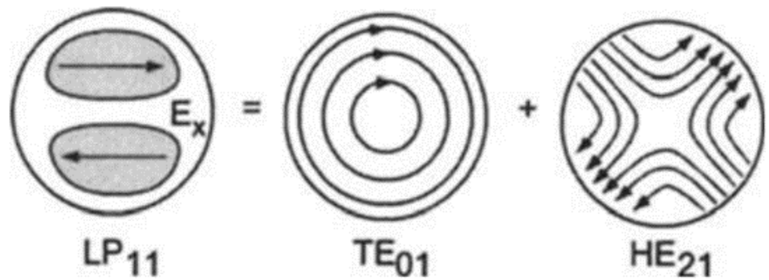
The plot shows $\hat{y}$ -direction, , but the mode could also be polarized along the $\hat{x}$ -axis. Also, the lobes could be rotated by 90°, making the null region lie along the $x = 0$.

Thus, there are four degenerate LP_{11} modes (two orientations of the lobes, each with two possible polarizations).

The LP modes have many practical advantages.

First, the LP modes provide an easy way to visualize the structure of the guided modes. Because most of the energy is stored in the transverse field of the LP mode, we can ignore the complications of energy stored in the longitudinal terms.

1.9 THE LINEARLY POLARIZED MODES



Second, the LP modes represent actual energy distributions that a polarized source would excite in a fiber. For example, a polarized laser uniformly illuminating the end of a step would create a linearly polarized transverse field on the input.

Finally, LP modes allow for a simplified characteristic equation that can be solved with straightforward numeric or graphical techniques.

The disadvantages of LP modes are due to the fact that they are not true modes but are in fact a superposition of slightly nondegenerate modes.

The individual EH, HE, TM, and TE modes travel at slightly different velocities, so the polarization state of the initial superposition will change as the modes propagate down the axis of the guide.

The LP modes are, in summary, only an approximation of the true mode structure of the fiber.

1.10 THE NORMALIZED FREQUENCY AND CUTOFF

Often, we are concerned whether a given mode will propagate within a fiber.

For example, we might need a single mode fiber for an experiment using a visible laser, such as the HeNe laser operating at $\lambda = 633\ nm$, but all we can find is single mode fiber that is designed for operation at 1.3 μm.

How can we determine if this fiber will be satisfactory?

To answer this, we need to develop what are known as "cut-off" conditions, which determine under what circumstances a mode will propagate in a fiber.

Let us consider the characteristic equation in its general form:

$$\frac{\beta^2 \upsilon^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2 = \left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\cdot\left[\frac{k_o^2 n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_o^2 n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

Considering also the Bessel identity:

$$J'_\nu(\kappa r) = \frac{1}{2}\left[J_{\upsilon-1}(\kappa r) - J_{\upsilon+1}(\kappa r)\right]$$

we can clearly see that the characteristic equation contains a term with the ratio of Bessel functions $\frac{J_{\upsilon\pm1}}{J_\nu}$

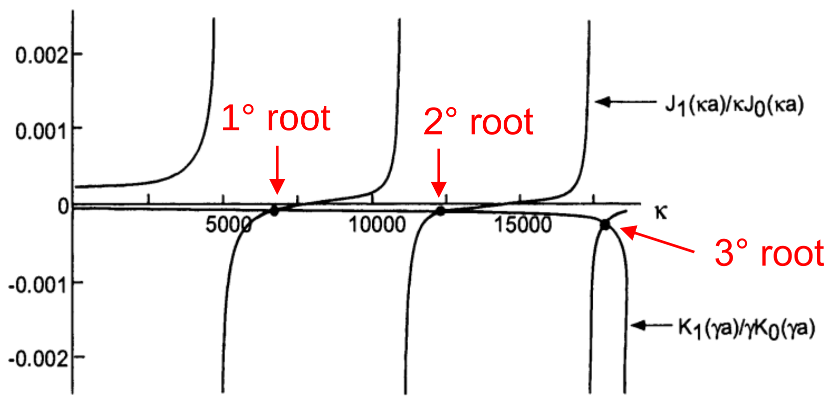
1.10 THE NORMALIZED FREQUENCY AND CUTOFF

$$\frac{\beta^2 \upsilon^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2=\left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\cdot\left[\frac{k_o^2 n_{core}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_o^2 n_{clad}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

$$J'_\nu(\kappa r)=\frac{1}{2}\left[J_{\upsilon-1}(\kappa r)-J_{\upsilon+1}(\kappa r)\right]$$

This term explodes to infinity at each root of J_υ, as the student will demonstrate in the homework.

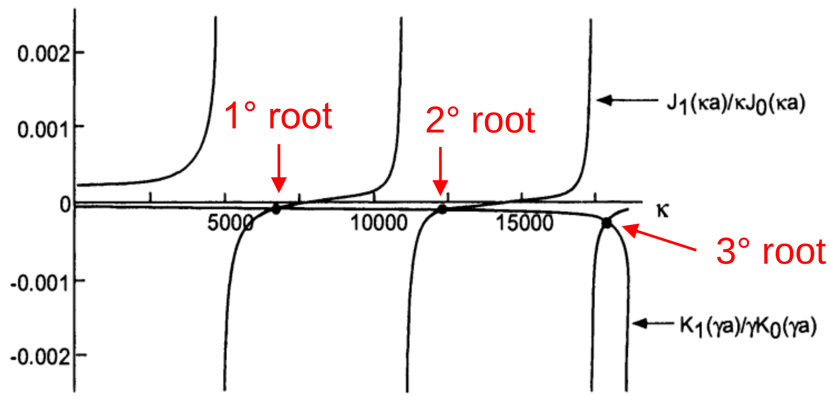
A representative trend is reported in the following picture:



$$\frac{J_1(\kappa a)}{\kappa J_0(\kappa a)}=-\frac{K_1(\kappa a)}{\gamma K_0(\gamma a)}$$

To ensure that there is at least one solution to the equation (i.e., one place where the lines cross), the argument κr must extend beyond the first root.

1.10 THE NORMALIZED FREQUENCY AND CUTOFF



$$\frac{J_1(\kappa a)}{\kappa J_0(\kappa a)} = -\frac{K_1(\kappa a)}{\gamma K_0(\gamma a)}$$

Each time κa increases beyond another root of $\frac{J_{v\pm1}}{J_v}$, another mode will be allowed.

The roots of the Bessel functions are thus the signposts for establishing mode cutoff conditions.

We can generalize the cutoff conditions for the modes in terms of the roots of the appropriate Bessel function.

For example, referring back to Picture, it is clear that no TE mode will exist if $\kappa a < 2.405$ (that is the first root of $J_0(\kappa a)$).

The TE_{01} can only exist if $\kappa a < 2.405$, so we say that the cut-off condition for the TE_{01} mode is $\kappa a = 2.405$.

1.10 THE NORMALIZED FREQUENCY AND CUTOFF

The cutoff condition for the TE_{02} mode occurs at the second root of the Bessel function, $J_0(\kappa a)$ which occurs 5.405.

The cutoff conditions for every variety of mode can be found in a similar fashion.

These cutoff conditions are

TE_{0m} modes $\rightarrow \kappa a > m^{th}$ root of $J_0(\kappa a)$

HE_{1m} modes $\rightarrow \kappa a > m^{th}$ root of $J_1(\kappa a)$

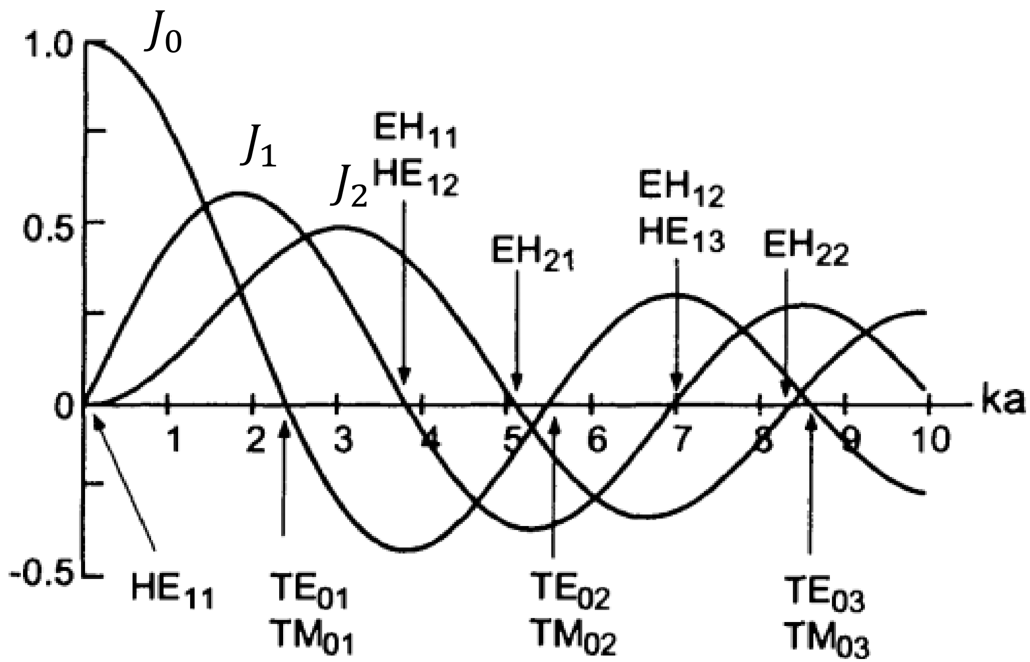
$EH_{\upsilon m}$ modes $\rightarrow \kappa a > m^{th}$ root of $J_\upsilon(\kappa a)$ with the added constraint that the first root is not 0

$HE_{\upsilon m}$ modes $\rightarrow \left(\frac{\epsilon_{core}}{\epsilon_{clad}} + 1\right) J_{\upsilon-1}(\kappa a) = \frac{\kappa a}{\upsilon - 1} J_\upsilon(\kappa a)$

Thus, the modes allowed in a circular waveguides can be easily found by plotting the Bessel functions $J_\upsilon(\kappa r)$ as a function of κa.

1.10 THE NORMALIZED FREQUENCY AND CUTOFF

Figure shows a plot of the first three Bessel functions, with notations on the cutoff points for a few modes.



For example, if κa is greater than 2.405, then the TE_{01}, TM_{01}, and HE_{21} modes will be allowed. This is in addition to the HE_{11} mode, which is always allowed.

The HE_{11} mode is a special case which will be described in the next chapter.

1.10 THE NORMALIZED FREQUENCY AND CUTOFF

The HE_{vm} modes have a complicated cutoff formula which requires knowledge of the refractive indices of the core and cladding.

In most cases the ratio can be approximated as unity and the cutoff condition

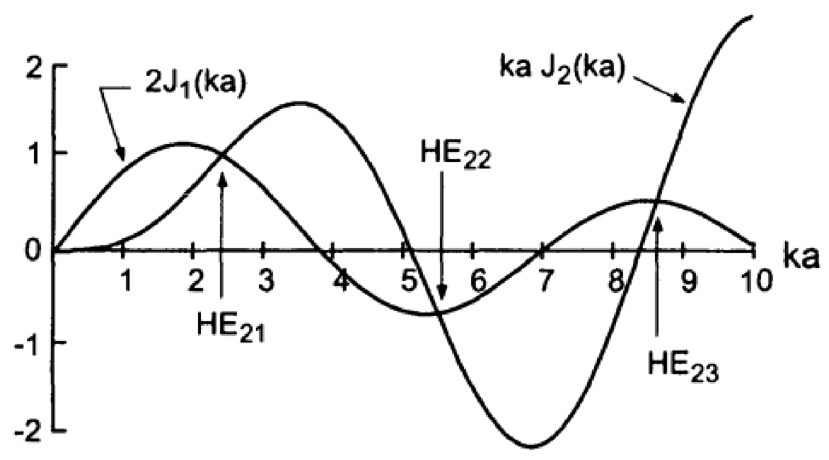
$$\left(\frac{\epsilon_{core}}{\epsilon_{clad}}+1\right)J_{\upsilon-1}(\kappa a)=\frac{\kappa a}{\upsilon-1}J_{\upsilon}(\kappa a)$$

becomes $$2J_{\upsilon-1}(\kappa a)=\frac{\kappa a}{\upsilon-1}J_{\upsilon}(\kappa a)$$

Figure shows the cut-off condition for the HE_{2m} modes.

Being $\upsilon = 2$ for HE_{2m} modes, the cutoff condition is $2J_1(\kappa a) = \kappa a J_2(\kappa a)$.

When $\kappa a > 2.405$, together with HE_{11}, TE_{01} and TM_{01} modes, also HE_{21} is allowed.



1.10 THE NORMALIZED FREQUENCY AND CUTOFF

The parameter used to characterize a waveguide is the *Normalized Frequency* or the V-number.

For a cylindrical fiber, the V-number is defined as

$$V - number = ak_{max} = ak_0\sqrt{n_{core}^2 - n_{clad}^2} = \frac{2\pi a}{\lambda}\sqrt{n_{core}^2 - n_{clad}^2}$$

where a is the core radius.

The normalized frequency provides a quick way to determine the number of modes in a waveguide and is often used as a specification for optical fibers and devices.

The cutoff conditions can all be evaluated once the V-number of a fiber is given.

The V-number is useful for determining cutoff conditions, as well as the total number of allowed modes.

The V -number is often specified in the purchase of optical single mode fiber.

For example, the cut-off condition for a single mode fiber occurs when the V -number reaches 2.405 (the first root of the J_0 Bessel function).

The term "cut-off' refers to the point where the TE_{01}, TM_{01}, and HE_{21} modes cease to propagate if V becomes smaller.

The wavelength at which a single-mode fiber suddenly becomes multimode is called the "cutoff' wavelength λ_c

1.11 TOTAL NUMBER OF MODES IN A STEP-INDEX WAVEGUIDE

For large core diameter fibers with many modes, it is possible to provide an approximate formula describing the total number of modes that will propagate.

Recall the characteristic equation for the LP modes

$$\kappa \frac{J_{j-1}(\kappa a)}{J_j(\kappa a)} = -\gamma \frac{K_{j-1}(\gamma a)}{K_j(\gamma a)}$$

For values of κa far from cutoff, the term γa will be large, and the asymptotic value of the K_j functions can be used.

$$K_j(\gamma a) \rightarrow \frac{e^{-\gamma a}}{\sqrt{2\pi\, \gamma a}}$$

Thus, the ratio $\frac{K_{j-1}(\gamma a)}{K_j(\gamma a)} \rightarrow 1$ and the characteristic equation reduces to:

$$\frac{J_{j-1}(\kappa a)}{J_j(\kappa a)} = -\frac{\gamma}{\kappa}$$

1.11 TOTAL NUMBER OF MODES IN A STEP-INDEX WAVEGUIDE

$$\frac{J_{j-1}(\kappa a)}{J_j(\kappa a)} = -\frac{\gamma}{\kappa}$$

For a given value of v, the number of allowed modes will be proportional to the number of roots of $J_j(\kappa a)$ between 0 and $\kappa a = V$.

In the approximation that κa is large, the asymptotic expansion of $J_j(\kappa a)$ can be used

$$J_j(\kappa a) \approx \sqrt{\frac{2}{\pi \kappa a}} cos\left(\kappa a - \frac{v\pi}{2} - \frac{\pi}{4}\right)$$

There will be one root every time the ratio goes to infinity i.e., each time the argument increases by π.

For a given value of v, the number of roots will be approximately

$$m = \frac{\kappa a - \frac{v\pi}{2} - \frac{\pi}{4}}{\pi}$$

1.11 TOTAL NUMBER OF MODES IN A STEP-INDEX WAVEGUIDE

$$m = \frac{\kappa a - \frac{\nu\pi}{2} - \frac{\pi}{4}}{\pi}$$

Solving this in terms of the normalized frequency, $V = \kappa_{max}a$, and ignoring the $\frac{\pi}{4}$ term

$$V = (2m + \upsilon)\frac{\pi}{2}$$

This equation, while only an approximation, shows the general relationship between the azimuthal number υ, and the number of radial nodes in the mode m.

As υ increases, indicating more angular lobes, the maximum value of m must decrease, implying that the radial structure becomes smoother.

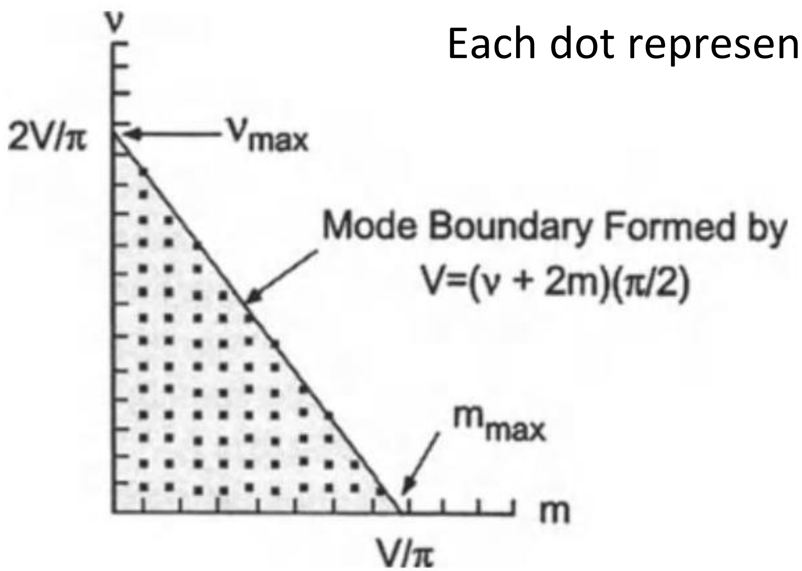
Since there is an allowed mode for each value of m and υ, we can graphically plot the number of modes.

The largest possible value for m is V/π, when $\upsilon = 0$.

Likewise, the maximum value for υ is $2V/\pi$.

These allowed values are plotted in the following Figure.

1.11 TOTAL NUMBER OF MODES IN A STEP-INDEX WAVEGUIDE



Each dot represents an allowed combination of m and v.

The total number of allowed modes is geometrically determined from the area of the triangle in the Figure:

$$\frac{m_{max}v_{max}}{2} = \frac{V^2}{\pi^2}$$

$$m_{max} = \frac{V}{\pi}$$

$$v_{max} = \frac{2V}{\pi}$$

We must recall that for each mode, there are two angular orientations *(cosine* or *sine* solution), and two possible polarizations ($\hat{x}$ or $\hat{y}$ in the LP mode approximation).

The number of modes is increased by a factor of four.

So, the number of allowed modes in a fiber waveguide is given by the approximation :

$$N = \frac{4V^2}{\pi^2}$$