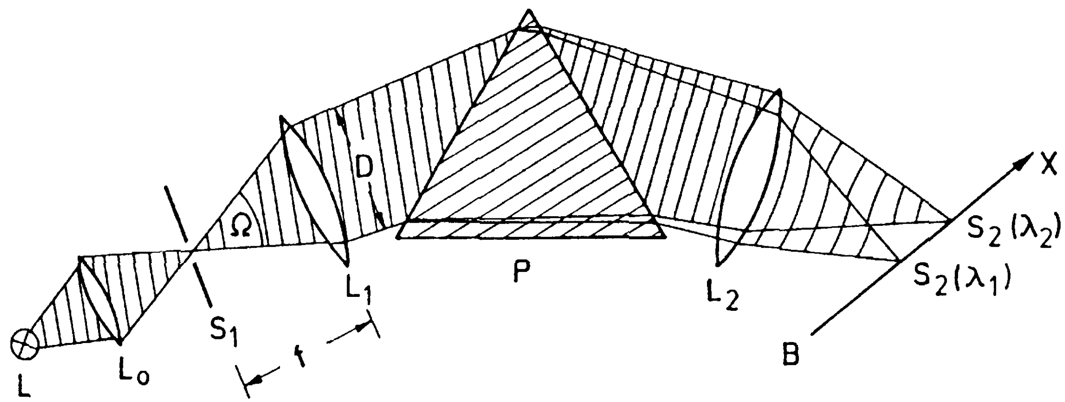
CHAPTER 4

SPECTROSCOPIC INSTRUMENTATION

4.1 SPECTROGRAPHS AND MONOCHROMATORS

Spectrographs are optical instruments that form an image $S_2(\lambda)$ of the entrance slit S_1; images are laterally separated for different wavelengths λ of the incident radiation. This lateral dispersion is due to either spectral dispersion in a prism or to diffraction on plane or reflection gratings.

Prism spectrograph



The light source L illuminates the entrance slit S_1, which is placed in the focal plane of the collimator lens L_1. Behind L_1 the parallel light beam passes through the prism P, where is diffracted by an angle $\theta(\lambda)$ depending on the wavelength λ. The lens L_2 form an image $S_2(\lambda)$ of the entrance slit S_1.

The position $x(\lambda)$ of the image in the focal plane of L_2 is a function of the wavelength λ. The linear dispersion $dx/d\lambda$ of the spectrograph depends on spectral dispersion $dn/d\lambda$ of the prism material and on the focal length L_2.

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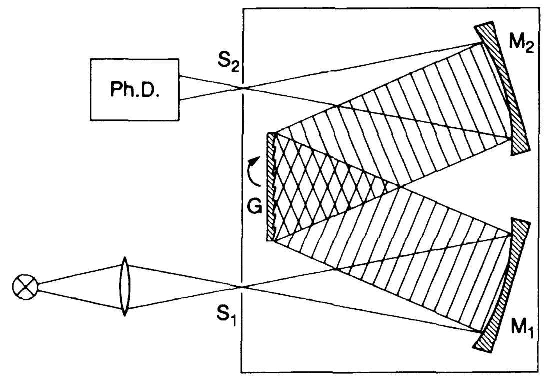
When a reflecting diffraction grating is used to separate the spectral lines $S_2(\lambda)$, the two lenses L_1 and L_2 are commonly replaced by two spherical mirrors M_1 and M_2, which image the entrance slit onto the plane of observation.

Grating spectrograph



Both systems can use either photographic or photoelectric recording. According to the kind of detection, we distinguish between **spectrographs** and **monochromators**.

In spectrographs a charge-coupled device (CCD) diode array is placed in the focal plane of L_2 or M_2. The whole spectral range $\Delta\lambda = \lambda_1(x_1) - \lambda_2(x_2)$ covered by the lateral extension $\Delta x = x_1 - x_2$ of the diode array can be recorded simultaneously. The spectral range is limited by the spectral sensitivity of available CCD materials.

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- Monochromators, on the other hand, use photoelectric recording of a selected small spectral interval.
- The exit slit S_2 selecting an interval Δx_2 in the focal plane B lets only the limited range $\Delta\lambda$ through to the photoelectric detector.
- Different spectral ranges can be detected by shifting S_2 along the lateral direction x.
- A more convenient solution turns the prism or grating by a gear-box drive, which allows the different spectral regions to be tuned across the fixed exit slit.

- Unlike the spectrograph, different spectral regions are not detected simulta-
- simultaneously but successively.
- The signal received by the detector is proportional to the product of the area $h\Delta x_2$ of the exit slit with height h with spectral intensity $\int I(\lambda)d\lambda$, where the integration extends over the spectral range dispersed within the width Δx_2 of S_2.

- In the literature, the term **spectrometer** is often used to refer to both instruments.

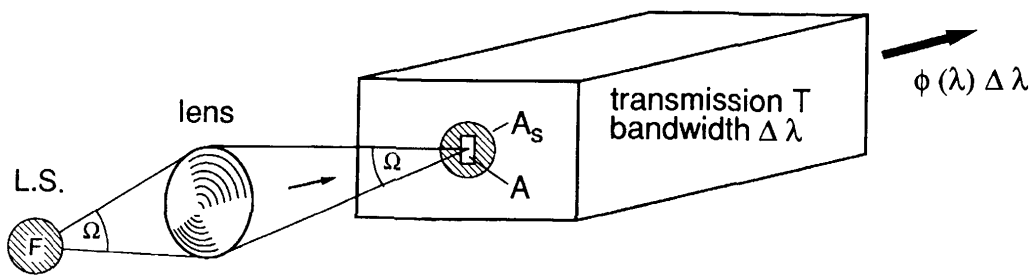
The selection of the optimum type of spectrometer for a particular experiment is guided by some basic characteristics of spectrometers and their relevance to the particular application .

4.1 SPECTROGRAPHS AND MONOCHROMATORS

4.1.1 Speed of a spectrometer

When the spectral intensity I_λ^* within the solid angle $d\Omega = 1\ sr$ is incident on the entrance slit of area A, a spectrometer with an acceptance angle Ω transmits the radiant flux within the spectral interval $d\lambda$:

$$\phi_\lambda d\lambda = I_\lambda^* \frac{A}{A_s} T(\lambda)\Omega d\lambda$$



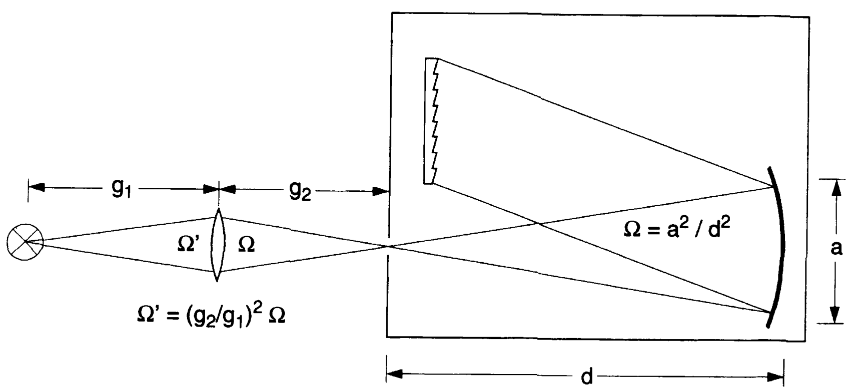
where $A_s \geq A$ is the area of the source image at the entrance slit and $T(\lambda)$ the transmission of the spectrometer.

The product $U = A\Omega$ is often named **étendue**. For the prism spectrograph the maximum solid angle of acceptance is limited by the effective area of the prisms; For the grating spectrometer the sizes of the grating and mirrors limit the acceptance solid angle Ω.

4.1 SPECTROGRAPHS AND MONOCHROMATORS

4.1.1 Speed of a spectrometer

In order to utilize the optimum speed, it is advantageous to image the light source onto the entrance slit in such a way that the acceptance angle Ω is fully used .



Optimized imaging of a light source onto the entrance slit of a spectrometer is achieved when the solid angle Ω' *of the incoming light matches the acceptance angle* $\Omega = \left(\frac{a}{d}\right)^2$ *of the spectrometer*

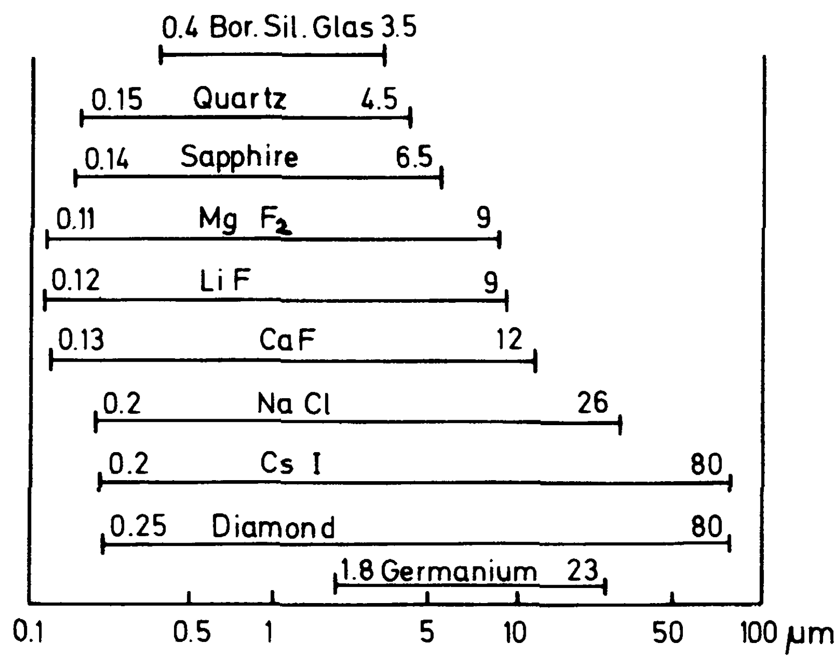
Although more radiant power from an extended source can pass the entrance slit by using a converging lens to reduce the source image on the entrance slit, the divergence is increased. The radiation outside the acceptance angle Ω cannot be detected, but may increase the background by scattering from lens holders and spectrometer walls. .

4.1 SPECTROGRAPHS AND MONOCHROMATORS

4.1.2 Spectral transmission

For prism spectrometers, the spectral transmission depends on the material of the prism and the lenses.

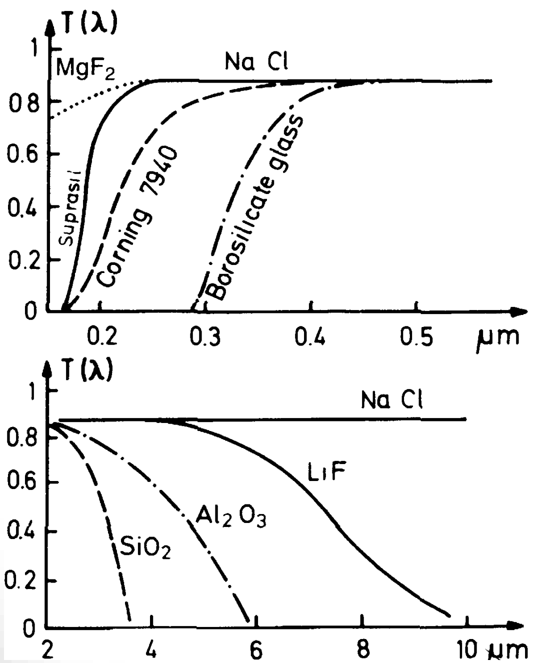
Using fused quartz, the accessible spectral range spans from about 180 to 3000 nm. Below 180nm (vacuum-ultraviolet region), the whole spectrograph must be evacuated, and lithium fluoride or calcium fluoride must be used for the prism and the lenses, although most VUV spectrometers are equipped with reflection gratings and mirrors.



a)

4.1 SPECTROGRAPHS AND MONOCHROMATORS

4.1.2 Spectral transmission



In the infrared region, several materials (for example, CaF2, NaCl, and KBr crystals) are transparent up to 30 µm.

However, because of the high reflectivity of metallic coated mirrors and gratings in the infrared region, grating spectrometers with mirrors are preferred over prism spectrographs.

Many vibrational-rotational transitions of molecules such as H_2O or CO_2 fall within the range 3—10 µm, causing selective absorption of the transmitted radiation. .

Infrared spectrometers therefore have to be either evacuated or filled with dry nitrogen.

4.1 SPECTROGRAPHS AND MONOCHROMATORS

4.1.3 Spectral resolving power

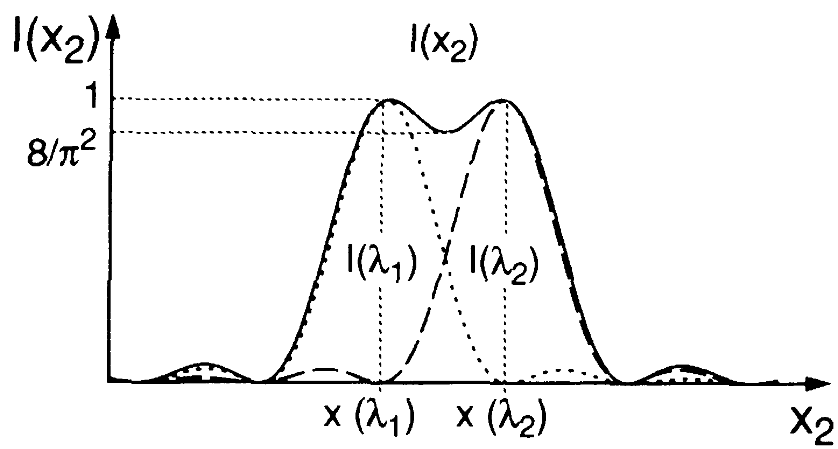
The spectral resolving power of any dispersing instrument is defined by the expression:

$$R = \left|\frac{\lambda}{\Delta\lambda}\right| = \left|\frac{\upsilon}{\Delta\upsilon}\right|$$

where $\Delta\lambda = \lambda_1 - \lambda_2$ stands for the minimum separation of the central wavelengths of two λ_1 e λ_2 of two closely spaced lines that are considered to be just resolved.

What is meant by resolved?

An intensity distribution composed of two lines with the intensity profiles $I_1(\lambda - \lambda_1)$ and $I_2(\lambda - \lambda_2)$ can be recognize if the total intensity $I(\lambda) = I_1(\lambda - \lambda_1) + I_2(\lambda - \lambda_2)$ shows a pronounced dip between two maxima.



4.1 SPECTROGRAPHS AND MONOCHROMATORS

4.1.3 Spectral resolving power

The intensity distribution $I(\lambda)$ depends on the ratio I_1/I_2 and on the profiles of both components. Therefore, the minimum resolvable interval $\Delta\lambda$ will differ for different profiles.

Lord Rayleigh introduced a criterion of resolution for diffraction-limited line profiles, where **two lines are considered to be just resolved if the central diffraction maximum of the profile $\boldsymbol{I_1(\lambda-\lambda_1)}$ coincides with the first minimum of $\boldsymbol{I_2(\lambda-\lambda_2)}$**.

Let us consider the attainable spectral resolving power of a spectrometer. When passing the dispersing element (prism or grating), a parallel beam composed of two monochromatic waves with wavelengths λ and $\lambda+\Delta\lambda$ is split into two partial beams with the angular deviations θ and $\theta+\Delta\theta$ from their initial direction. The angular separation is:

$$\Delta\theta = \frac{d\theta}{d\lambda}\Delta\lambda$$

where $\frac{d\theta}{d\lambda}$ is called the angular dispersion [rad/nm].

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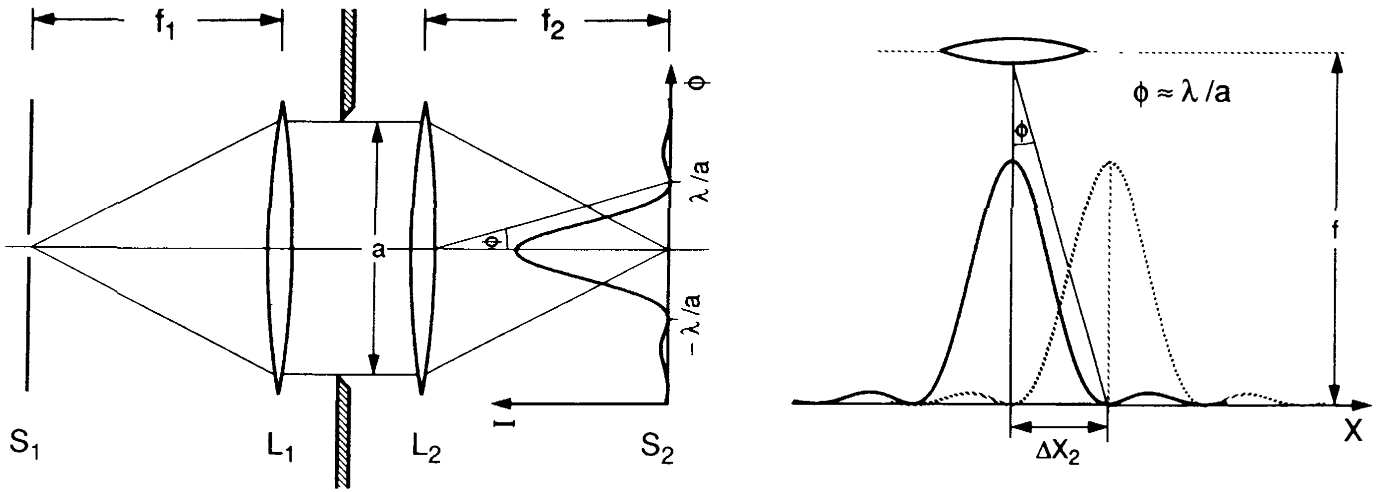
4.1.3 Spectral resolving power



Since the camera lens with focal length f_2 images the entrance slit S_1 into the plane B, the distance Δx_2 between the two images $S_2(\lambda)$ and $S_2(\lambda + \Delta\lambda)$ is:

$$\Delta x_2 = f_2 \Delta\theta = f_2 \frac{d\theta}{d\lambda}\Delta\lambda = \frac{dx}{d\lambda}\Delta\lambda$$

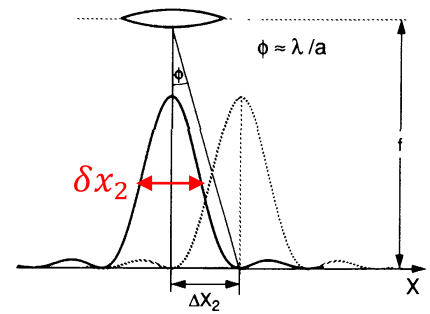
The factor $\frac{dx}{d\lambda}$ is called the linear dispersion of the instrument [mm/nm].

4.1 SPECTROGRAPHS AND MONOCHROMATORS

4.1.3 Spectral resolving power

In order to resolve two lines at λ and $\lambda + \Delta\lambda$, the separation Δx_2 has to be at least the sum $[\delta x_2(\lambda) + \delta x_2(\lambda + \Delta\lambda)]/2$ of the widths of the two slit images.
Since the width δx_2 is related to the width δx_1 of the entrance slit according to geometrical optics by:

$$\delta x_2 = \frac{f_2}{f_1}\delta x_1$$



the resolving power $\lambda/\Delta\lambda$ can be increased by decreasing δx_1.

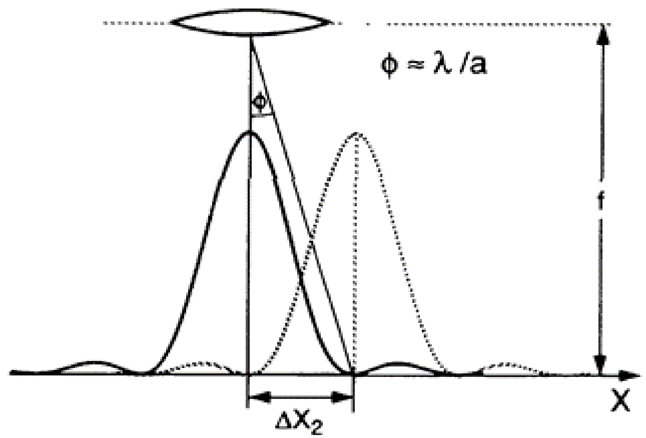
Unfortunately, there is a theoretical limitation set by diffraction. Because of the fundamental. When a parallel light beam passes a limiting aperture with diameter a, a Fraunhofer diffraction pattern is produced in the plane of the focusing L_2 lens.

The intensity distribution $I(\phi)$ as a function of the angle ϕ with the optical axis of the system is given by the well-known formula :

$$I(\phi) = I_0 \left[\frac{sen\left(\frac{a\pi sen\phi}{\lambda}\right)}{\left(\frac{a\pi sen\phi}{\lambda}\right)}\right]^2 \simeq I_0 \left[\frac{sen\left(\frac{a\pi\phi}{\lambda}\right)}{\left(\frac{a\pi\phi}{\lambda}\right)}\right]^2$$

4.1 SPECTROGRAPHS AND MONOCHROMATORS

4.1.3 Spectral resolving power



The first two diffraction minima at $\phi = \pm\frac{\lambda}{a} \ll \pi$ are symmetrical to the central maximum (zeroth diffraction order) at $\phi = 0$. The central maximum contains about 90% of the total intensity. .

Even an infinitesimally small entrance slit therefore produces a slit image of width :

$$\delta {x_2}^{diffr} = f_2 \frac{\lambda}{a}$$

defined as the distance between the central diffraction maximum and the first minimum, which is approximately equal to the FWHM of the central maximum.

4.1 SPECTROGRAPHS AND MONOCHROMATORS

4.1.3 Spectral resolving power

According to the Rayleigh criterion, two equally intense spectral lines with wavelengths λ and $\lambda + \Delta\lambda$ are just resolved if the central diffraction maximum of $S_2(\lambda)$ coincides with the first minimum of $S_2(\lambda + \Delta\lambda)$.

This means that their maxima are just separated by $\delta x_2{}^{diffr} = \Delta x_2 = f_2 \frac{\lambda}{a}$.

Combining this expression with $\Delta x_2 = f_2 \frac{d\theta}{d\lambda} \Delta\lambda$, the fundamental limit on the resolving power is determined:

$$\left| \frac{\lambda}{\Delta\lambda} \right| \leq a \frac{d\theta}{d\lambda}$$

which clearly depends only on the size a of the limiting aperture and on the angular dispersion of the instrument.

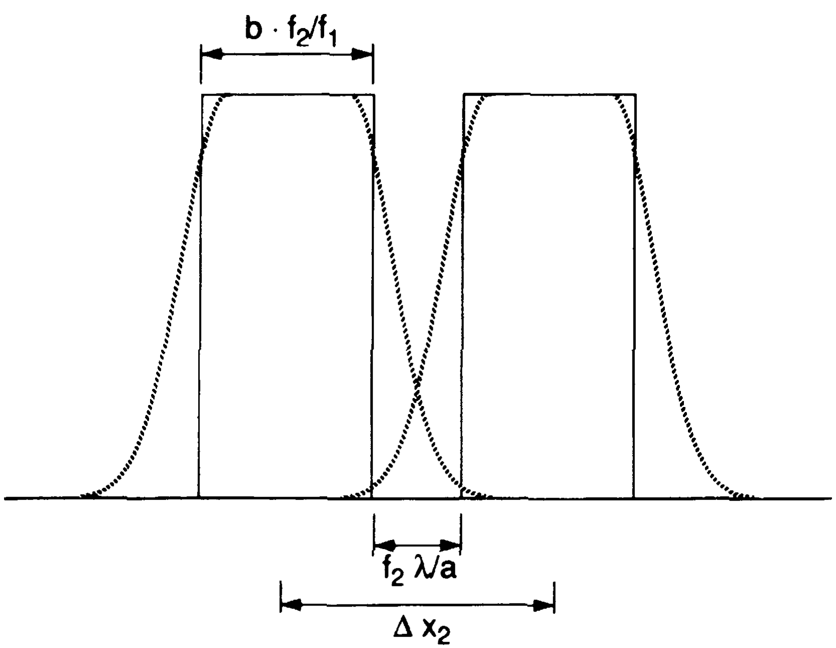
4.1 SPECTROGRAPHS AND MONOCHROMATORS

4.1.3 Spectral resolving power

For a finite entrance slit with width b, the separation Δx_2 between the central peaks of the two images $I_1(\lambda - \lambda_1)$ and $I_2(\lambda - \lambda_2)$ must be:

$$\Delta x_2 \geq f_2 \frac{\lambda}{a} + b\frac{f_2}{f_1}$$

in order to meet the Rayleigh criterion.



Intensity profiles of two monochromatic lines measured in the focal plane of L_2 with an entrance slit width $b \gg f_1\lambda/a$ and a magnification factor f_2/f_1 of the spectrograph.

Solid line: without diffraction;

Dashed line: with diffraction.

4.1 SPECTROGRAPHS AND MONOCHROMATORS

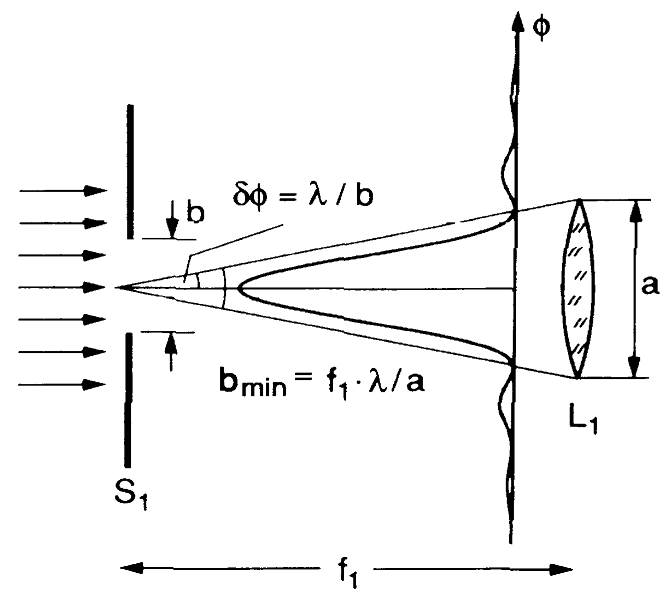
4.1.3 Spectral resolving power

With $\Delta x_2 = f_2 \frac{d\theta}{d\lambda} \Delta\lambda$, the smallest resolvable wavelength interval:

$$\Delta\lambda \geq \left(\frac{\lambda}{a} + \frac{b}{f_1}\right)\left(\frac{d\theta}{d\lambda}\right)^{-1}$$

Although it does not influence the spectral resolution, the much larger diffraction by the entrance slit imposes a limitation on the transmitted intensity at small slit widths.

This can be seen as follows: when illuminated with parallel light, the entrance slit with width b produces a Fraunhofer diffraction pattern analogous with a replaced by b.



Diffraction by the entrance slit

4.1 SPECTROGRAPHS AND MONOCHROMATORS

4.1.3 Spectral resolving power

The central diffraction maximum extends between angles$\delta\phi = \pm\frac{\lambda}{b}$ and can completely pass the limiting aperture a only if $2\delta\phi$ is smaller than the acceptance angle $\frac{a}{f_1}$ of the spectrometer. This imposes a lower limit to the useful width b_{min} of the entrance slit:

$$b_{min} \geq 2\lambda\frac{f_1}{a}$$

Replacing $b = b_{min} = 2\lambda\frac{f_1}{a}$ in $\Delta\lambda \geq \left(\frac{\lambda}{a} + \frac{b}{f_1}\right)\left(\frac{d\theta}{d\lambda}\right)^{-1}$ yields the practical limit for $\Delta\lambda$ imposed by diffraction by S_1 and by the limiting aperture with width a:

$$\Delta\lambda = 3f\frac{\lambda}{a}\frac{d\lambda}{dx}$$

Instead of the theoretical limit $\left|\frac{\lambda}{\Delta\lambda}\right| \leq a\frac{d\theta}{d\lambda}$ given by the diffraction through the aperture a, a smaller practically attainable resolving power is obtained from the last expression which takes into account a finite minimum entrance slit width b_{min} imposed by intensity considerations and which yields: :

$$R = \frac{\lambda}{\Delta\lambda} = \frac{a}{3}\frac{d\theta}{d\lambda}$$

4.1 SPECTROGRAPHS AND MONOCHROMATORS

4.1.4 Free Spectral Range

The free spectral range of a spectrometer is the wavelength interval $\delta\lambda$ of the incident radiation for which a one-valued relation exists between λ and the position $x(\lambda)$ of the entrance slit image.

Two spectral lines with wavelengths $\lambda_1$and $\lambda_2 = \lambda_1 \pm \delta\lambda$ cannot be distinguished without further information. This means that the wavelength λ measured by the instrument must be known beforehand with an uncertainty $\Delta\lambda < \delta\lambda$.

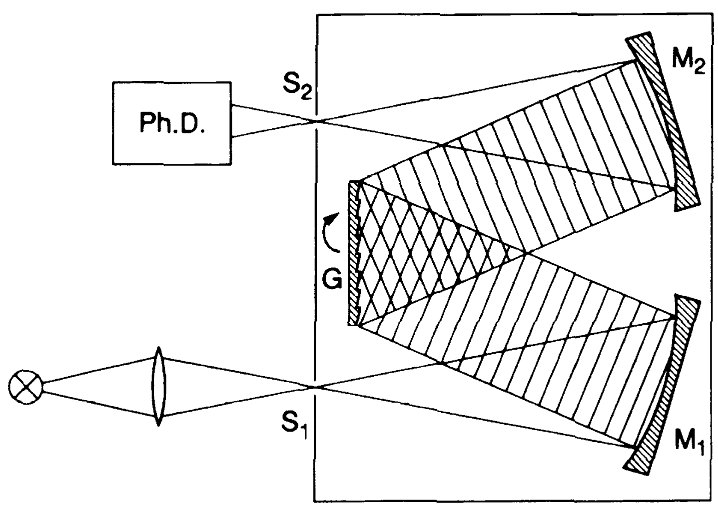
While for prism spectrometers the free spectral range covers the whole region of normal dispersion of the prism material, for grating spectrometers $\delta\lambda$ is determined by the diffraction order m and decreases with increasing m.
Interferometers, which are generally used in very high orders $(m = 10^4 - 10^8)$, have a high spectral resolution but a small free spectral range $\delta\lambda$.
For unambiguous wavelength determination they need a preselector, which allows one to measure the wavelength within $\delta\lambda$ of the high-resolution instrument.

4.2 GRATING SPECTROMETER

In a grating spectrometer the collimating lens L_1 is replaced by a spherical mirror M_1 with the entrance slit S_1 in the focal plane of M_1.

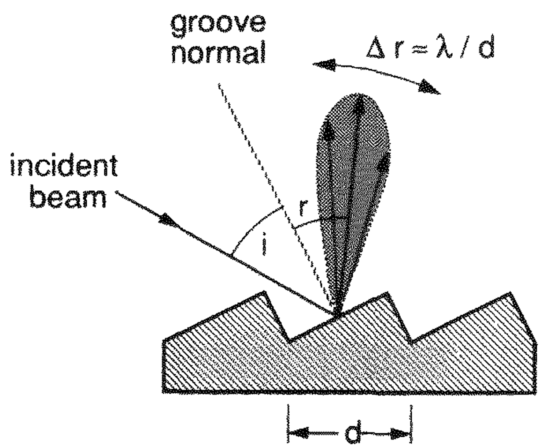
The collimated parallel light is reflected by M_1 onto a reflection grating consisting of many straight grooves (about 10^5) parallel to the entrance slit. The grooves have been ruled onto an optically smooth glass substrate or have been produced by holographic techniques.

The whole grating surface is coated with a highly reflecting layer (metal or dielectric film). The light reflected from the grating is focused by the spherical mirror M_2 onto the exit S_2 or onto a photographic plate in the focal plane of M_2.



4.2 GRATING SPECTROMETER

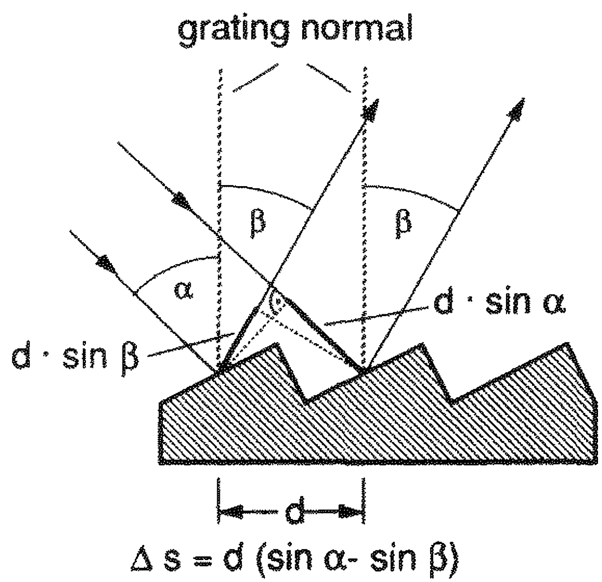
The many grooves, which are illuminated coherently, can be regarded as small radiation sources, each of them diffracting the light incident onto this small groove with a width $d \approx \lambda$ into a large range $\Delta r \approx \frac{\lambda}{d}$ of angles r around the direction of geometrical reflection .



The total reflected light consists of a coherent superposition of these many partial contributions. Only in those directions where all partial waves emitted from the different grooves are in phase will constructive interference result in a large total intensity, while in all other directions the different contributions cancel by destructive interference.

4.2 GRATING SPECTROMETER

Consider a parallel light beam incident onto two adjacent grooves.



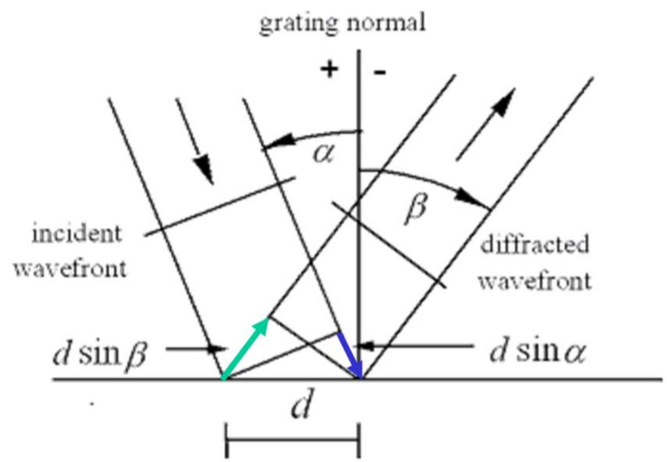
At an angle of incidence α to the grating normal (which is normal to

the grating surface, but not necessarily to the grooves) one obtains constructive interference for those directions β of the reflected light for which the path difference $\Delta s = \Delta s_1 - \Delta s_2$ is an integer multiple m of the wavelength λ. This yields the grating equation:

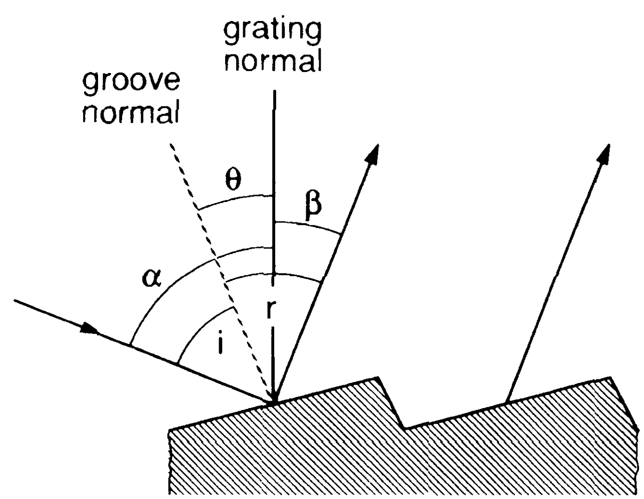
$$d(sen\alpha \pm sen\beta) = m\lambda$$

4.2 GRATING SPECTROMETER

the plus sign has to be taken if β and α are on the same side of the grating normal; otherwise the minus sign.

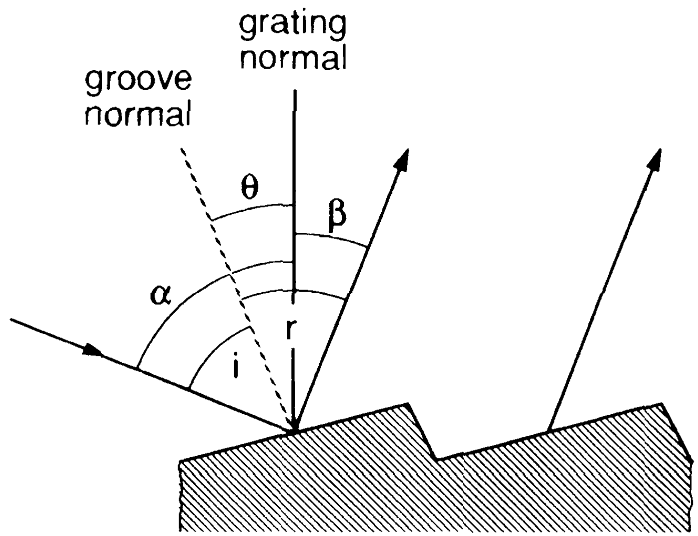


The reflectivity $R(\beta, \theta)$ of a ruled grating depends on the diffraction angle β and and on the blaze angle θ of the grating, which is the angle between the groove normal and the grating normal:



4.2 GRATING SPECTROMETER

If the diffraction angle β coincides with the angle r of specular reflection from the groove surfaces, $R(\beta,\theta)$ reaches its optimum value R_0, which depends on the reflectivity of the groove coating.



One infers for the case where α and β are on opposite sides of the grating normal, $i = \alpha - \theta$ e $r = \theta + \beta$, which yields, for specular reflection $i = r$, the condition for the optimum blaze angle θ:

$$\theta = \frac{\alpha - \beta}{2}$$

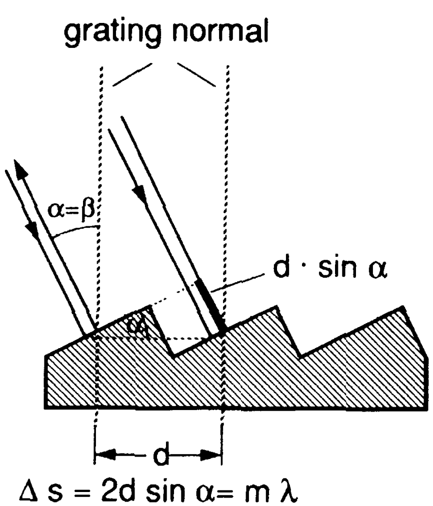
4.2 GRATING SPECTROMETER

Because of the diffraction of each partial wave into a large angular range, the reflectivity $R(\beta)$ will not have a sharp maximum at $\beta = \alpha - 2\theta$, but will rather show a broad distribution around this optimum angle.

The angle of incidence α is determined by the particular construction of the spectrometer, while the angle β for which constructive interference occurs depends on the wavelength λ.

Therefore, the blaze angle θ has to be specified for the desired spectral range and the spectrometer type.

In laser-spectroscopic applications the case $\alpha = \beta$, often occurs, which means that the light is reflected back into the direction of the incident light. Such an arrangement is called a **Littrow-grating mount**.



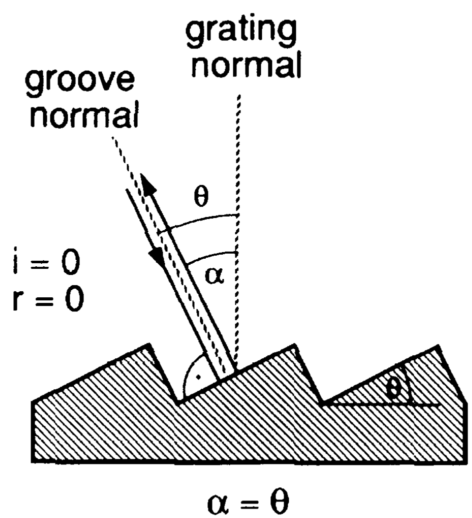
4.2 GRATING SPECTROMETER

The grating equation for constructive interference reduces to :

$$2dsen\alpha = m\lambda$$

$d(sen\alpha \pm sen\beta) = m\lambda$

Maximum reflectivity of the Littrow grating is achieved for $i = r = 0$, leading to $\theta = \alpha$ as shown in Figure.



Blaze angle for a Littrow grating

The Littrow grating acts as a wavelength-selective reflector because light is only reflected if the incident wavelength satisfies the coindition $2dsen\alpha = m\lambda$.

4.2 GRATING SPECTROMETER

We now examine the intensity distribution $I(\beta)$ of the reflected light when a monochromatic plane wave is incident onto an arbitrary grating.

The path difference between partial waves reflected by adjacent grooves is $\Delta s = d(sen\alpha \pm sen\beta)$ and the corresponding phase difference is:

$$\phi = \frac{2\pi}{\lambda}\Delta s = \frac{2\pi}{\lambda} d(sen\alpha \pm sen\beta)$$

The superposition of the amplitudes reflected from all N grooves in the direction β gives the total reflected amplitude:

$$A_R = \sqrt{R(\beta)} \sum_{m=0}^{N-1} A_g e^{im\phi} = \sqrt{R(\beta)} A_g \sum_{m=0}^{N-1} e^{im\phi}$$

In order to determine the amplitude, it is necessary to expand the series:

$$\sum_{m=0}^{N-1} e^{im\phi} = 1 + e^{i\phi} + e^{2i\phi} + .. + e^{i(N-1)\phi}$$

4.2 GRATING SPECTROMETER

We multiply both sides by $(1 - e^{i\phi})$:

$$\left(1 - e^{i\phi}\right) \sum_{m=0}^{N-1} e^{im\phi} = (1 - e^{i\phi})(1 + e^{i\phi} + e^{2i\phi} + .. + e^{i(N-1)\phi})$$

Expanding the terms of the product of the right-hand side, all the terms are eliminated each other, apart from 1 and $e^{iN\phi}$. Thus:

$$\sum_{m=0}^{N-1} e^{im\phi} = \frac{1 + e^{iN\phi}}{1 - e^{i\phi}}$$

Then substituting:

$$A_R = \sqrt{R(\beta)} A_g \frac{1 - e^{iN\phi}}{1 - e^{i\phi}}$$

where $R(\beta)$ is the reflectivity of the grating, which depends on the reflection angle β and A_g is the amplitude of the partial wave incident onto each groove.

4.2 GRATING SPECTROMETER

Because the intensity of the reflected wave is related to its amplitude by the relation:

$$I_R = \varepsilon_0 c A_R A_R^*$$

we have that the intensity of the reflected wave will be:

$$I_R = \varepsilon_0 c R(\beta)\, A_g A_g^* \left|\frac{1 - e^{iN\phi}}{1 - e^{i\phi}}\right|^2$$

Since that the square modulus of a ratio is equal to the ratio of the square modulus:

$$\left|\frac{1 - e^{iN\phi}}{1 - e^{i\phi}}\right|^2 = \frac{\left|1 - e^{iN\phi}\right|^2}{\left|1 - e^{i\phi}\right|^2} = \frac{|1 - cos(N\phi) - isen(N\phi)|^2}{|1 - cos(\phi) - isen(\phi)|^2}$$

$$= \frac{[1 - cos(N\phi)]^2 + sen^2(N\phi)}{[1 - cos(\phi)]^2 + sen^2(\phi)}$$

$$= \frac{1 + cos^2(N\phi) - 2cos(N\phi) + sen^2(N\phi)}{1 + cos^2(\phi) - 2cos(\phi) + sen^2(\phi)} = \frac{2 - 2cos(N\phi)}{2 - 2cos(\phi)}$$

4.2 GRATING SPECTROMETER

Since:

$$sen^2\left(\frac{\phi}{2}\right) = \frac{1 - cos\phi}{2}$$

we have:

$$\left|\frac{1 - e^{iN\phi}}{1 - e^{i\phi}}\right|^2 = \frac{sen^2\left(\frac{N\phi}{2}\right)}{sen^2\left(\frac{\phi}{2}\right)}$$

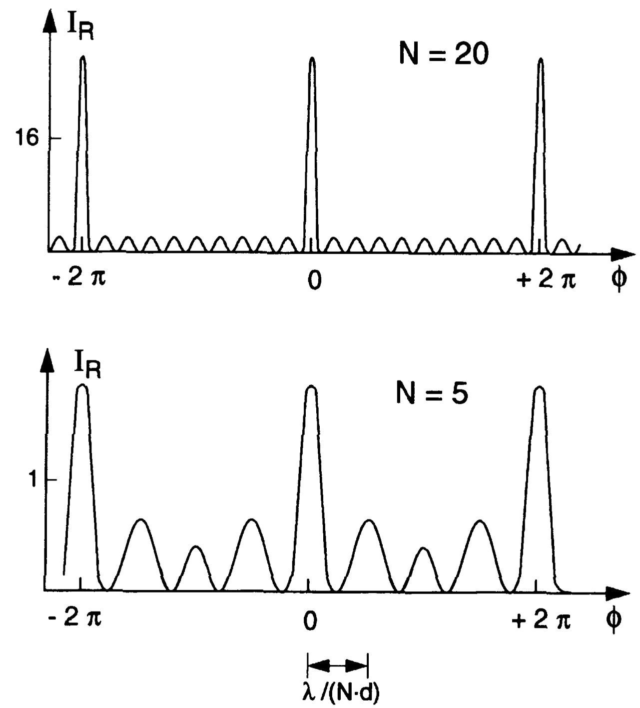
Then the intensity of the reflected wave can be expressed as:

$$I_R = I_0 R(\beta)\,\frac{sen^2\left(\frac{N\phi}{2}\right)}{sen^2\left(\frac{\phi}{2}\right)}$$

with $I_0 = \varepsilon_0 c A_g A_g^*$

4.2 GRATING SPECTROMETER

This intensity distribution is plotted in Figure for two different values of the total groove number N.



The principal maxima occur for $\phi = 2m\pi$, and the integer m is called the order of interference.

The function I_R has $N-1$ minima with $I_R = 0$ between two successive principal maxima. These minima occur at values of ϕ for which $N\frac{\phi}{2} = l\pi$, with $l = 1, 2, \ldots, N-1$

4.2 GRATING SPECTROMETER

The line profile $I(\beta)$ of the principal maximum of order m m around the diffraction angle β_m can be derived by substituting $\beta = \beta_m + \varepsilon$ using the expression found for I_R.

$$I_R = I_0 R(\beta)\,\frac{sen^2\left(\frac{N\phi}{2}\right)}{sen^2\left(\frac{\phi}{2}\right)}$$

$$\phi = \frac{2\pi}{\lambda} d(sen\alpha \pm sen\beta)$$

Because for large N, $I(\beta)$ is very sharply centered around β_m and we can assume $\varepsilon \ll \beta_m$.

Using the relation:

$$sen(\beta_m + \varepsilon) = sen\beta_m cos\varepsilon + cos\beta_m sen\varepsilon \sim sen\beta_m + \varepsilon cos\beta_m$$

in the expression for ϕ

$$\phi(\beta) = \frac{2\pi}{\lambda} d[sen\alpha \pm sen(\beta_m + \varepsilon)] = \frac{2\pi}{\lambda} d[sen\alpha \pm sen\beta_m + \varepsilon cos\beta_m]$$

from which:

$$\phi(\beta) = \overbrace{\frac{2\pi}{\lambda} d[sen\alpha \pm sen\beta_m]}^{2m\pi} + \frac{2\pi}{\lambda} d\varepsilon cos\beta_m = 2m\pi + \delta_1$$

assuming $\delta_1 = \frac{2\pi}{\lambda} d\varepsilon cos\beta_m \ll 1.$

4.2 GRATING SPECTROMETER

Replacing it in I_R, you get:

$$I_R = I_0 R(\beta)\,\frac{sen^2\left(\frac{N\delta_1}{2}\right)}{sen^2\left(\frac{\delta_1}{2}\right)} \simeq I_0 R(\beta)\,N^2\,\frac{sen^2\left(\frac{N\delta_1}{2}\right)}{\left(\frac{N\delta_1}{2}\right)^2}$$

$$\phi(\beta) = 2m\pi + \delta_1$$

$$I_R = I_0 R(\beta)\,\frac{sen^2\left(\frac{N\phi}{2}\right)}{sen^2\left(\frac{\phi}{2}\right)}$$

The first two minima on both sides of the central maximum at β_m are at:

$$N\delta_1 = \pm 2\pi$$

$$N\frac{\phi}{2} = l\pi$$

che corrisponde a:

$$\varepsilon_{1/2} = \frac{\pm\lambda}{Nd\cos\beta_m}$$

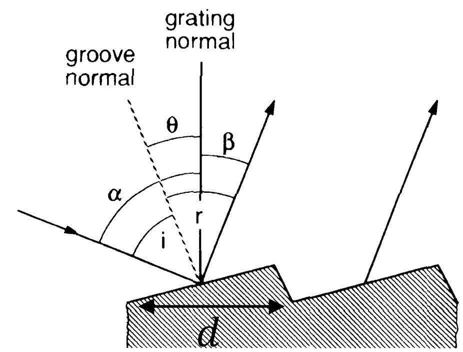
$$\delta_1 = \frac{2\pi}{\lambda} d\varepsilon\cos\beta_m$$

The central maximum of m-th order therefore has a line profile with a base halfwidth:

$$\Delta\beta = \frac{\lambda}{Nd\cos\beta_m}$$

4.2 GRATING SPECTROMETER

$$\Delta\beta = \frac{\lambda}{Ndcos\beta_m}$$



This corresponds to a diffraction pattern produced by an aperture with width $b = Ndcos\beta_m$, which is just the size of the whole grating projected onto a plane, normal to the direction normal of β_m:

Calculate the spectral resolving power. Differentiating the grating equation $d(sen\alpha \pm sen\beta) = m\lambda$ with respect to λ, we obtain at a given angle α the angular dispersion:

$$\frac{d\beta}{d\lambda} = \frac{m}{dcos\beta}$$

Combining it with the relation: $d(sen\alpha \pm sen\beta)$= $m\lambda$, you get:

$$\frac{d\beta}{d\lambda} = \frac{sen\alpha \pm sen\beta}{\lambda cos\beta}$$

This illustrates that the angular dispersion is determined solely by the angles α and β and not by the number of grooves!

4.2 GRATING SPECTROMETER

For the Littrow mount with $\alpha = \beta$, we obtain:

$$\frac{d\beta}{d\lambda} = \frac{2tg\alpha}{\lambda}$$

$$\frac{d\beta}{d\lambda} = \frac{sen\alpha \pm sen\beta}{\lambda cos\beta}$$

The resolving power can be immediately derived from its definition $\mathrm{R} = \frac{\lambda}{\Delta\lambda} \leq a\frac{d\theta}{d\lambda}$ and considering $a = Ndcos\beta$ as the size of the grating and $\frac{d\theta}{d\lambda} = \frac{d\beta}{d\lambda}$:

$$\frac{d\beta}{d\lambda}\Delta\lambda = \frac{\lambda}{Ndcos\beta}$$

Using the relation $\frac{d\beta}{d\lambda} = \frac{sen\alpha \pm sen\beta}{\lambda cos\beta}$, the latter can be rewritten as:

$$\frac{\lambda}{\Delta\lambda} = \frac{Nd(sen\alpha \pm sen\beta)}{\lambda}$$

which can be reduced, using $d(sen\alpha \pm sen\beta) = m\lambda$, as:

$$R = \frac{\lambda}{\Delta\lambda} = mN$$

4.2 GRATING SPECTROMETER

The theoretical spectral resolving power is the product of the diffraction order m with the total number N of illuminated grooves. If the finite slit width b and the diffraction at limiting apertures are taken into account, the practically achievable resolving power is about 2—3 times lower.

Often it is advantageous to use the spectrometer in second order (m = 2), which increases the spectral resolution by a factor of 2 without losing much intensity, if the blaze angle is correctly chosen to satisfy the condition with m = 2.

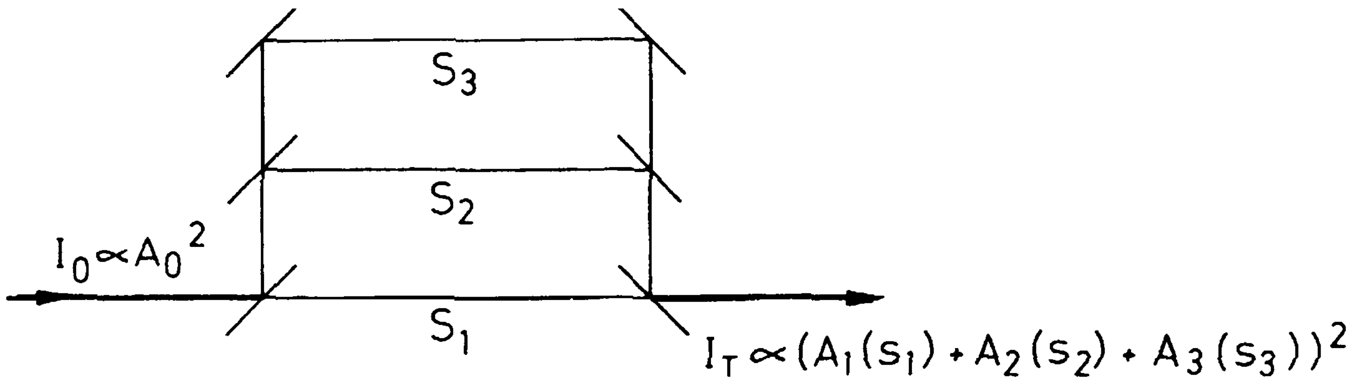
Minute deviations of the distance d between adjacent grooves, caused by inaccuracies during the ruling process, may result in constructive interference from parts of the grating for "wrong" wavelengths. Such unwanted maxima, which occur for a given angle of incidence a into "wrong" directions β, are called grating ghosts.

Although the intensity of these ghosts is generally very small, intense incident radiation at a wavelength λ may cause ghosts with intensities comparable to those of other weak lines in the spectrum. This problem is particularly serious in laser spectroscopy when the intense light at the laser wavelength, which is scattered by cell walls or windows, reaches the entrance slit of the monochromator.

4.3 INTERFEROMETERS

4.3.1 Basic concepts

The basic principle of all interferometers may be summarized as follows.



The incident lightwave with intensity I_0 is divided into two or more partial beams with amplitudes A_k which pass different optical path lengths $s_k = nx_k$ (with n is the refractive index), before they are again superimposed at the exit of the interferometer.

Since all partial beams come from the same source, they are coherent as long as the maximum path difference does not exceed the coherence length.

4.3 INTERFEROMETERS

4.3.1 Basic concepts

The total amplitude of the transmitted wave, which is the superposition of all partial waves, depends on the amplitudes A_k and on the phases $\phi_k = \phi_0 + \frac{2\pi s_k}{\lambda}$ of the partial waves. It is therefore sensitively dependent on the wavelength.

The maximum transmitted intensity is obtained when all partial waves interfere constructively. This gives the condition for the optical path difference $\Delta s_{ik} = s_i - s_k$ namely:

$$\Delta s_{ik} = m\lambda, \text{ con } m = 1,2,3 \ldots$$

Suppose that Δs_{ik} is fixed. The wavelengths matching the condition $\Delta s_{ik} = m\lambda$ will have the same phase shift:

The wavelength interval:

$$\delta\lambda = \lambda_m - \lambda_{m+1} = \frac{\Delta s}{m} - \frac{\Delta s}{m+1} = \frac{\Delta s}{m^2 + m} = \frac{\lambda}{m+1}$$

with $\lambda = \frac{\lambda_m + \lambda_{m+1}}{2}$ is called the **free spectral range** of the interferometer.

4.3 INTERFEROMETERS

4.3.1 Basic concepts

It is more conveniently expressed in terms of frequency. With $\upsilon = \frac{c}{\lambda}$, yields $\Delta s = \frac{mc}{\upsilon_m}$ and the free spectral frequency range:

$$\delta\upsilon = \upsilon_{m+1} - \upsilon_m = \frac{c}{\Delta s}$$

becomes independent of the order m.

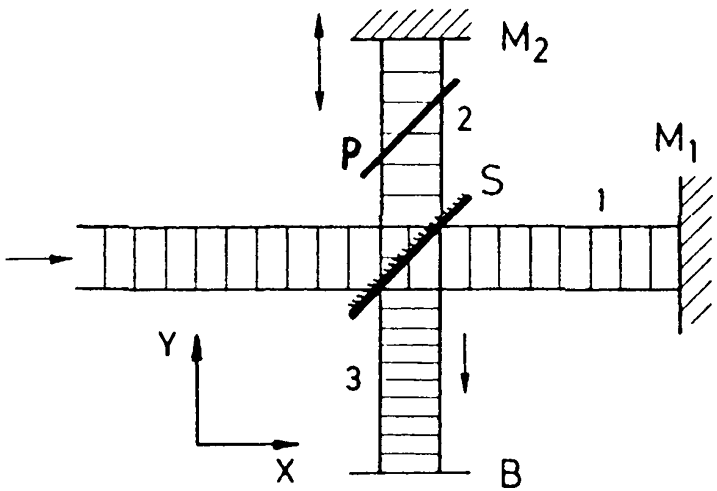
It is important to realize that from one interferometric measurement all wavelengths $\lambda = \lambda_0 + m\delta\lambda$ are equivalent with respect to the transmission of the interferometer. One therefore has at first to measure λ within one free spectral range using other techniques before the absolute wavelength can be obtained with an interferometer.

Examples of devices in which only two partial beams interfere are the Michelson interferometer and the Mach–Zehnder interferometer. Multiple-beam interference is used, for instance, in the grating spectrometer, the Fabry–Perot interferometer, and in multilayer dielectric coatings of highly reflecting mirrors.

4.3 INTERFEROMETERS

4.3.2 Michelson interferometer

The basic principle of the Michelson interferometer (MI) is illustrated in Figure:



The incident plane wave

$$E = A_0 e^{i(\omega t - kx)}$$

is split by the beam splitter S (with reflectivity R and transmittance T) into two waves:

$$E_1 = A_1 e^{i(\omega t - kx + \phi_1)}$$

$$E_2 = A_2 e^{i(\omega t - ky + \phi_2)}$$

4.3 INTERFEROMETERS

4.3.2 Michelson interferometer

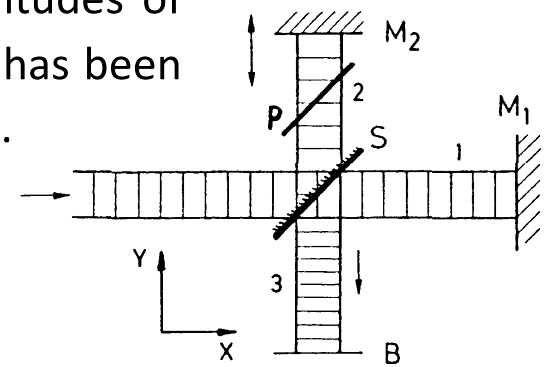
If the beam splitter has negligible absorption, $R + T = 1$, the amplitudes A_1 and A_2 are determined by:

$$\begin{cases} A_1 = \sqrt{R} A_0 \\ A_0^2 = A_1^2 + A_2^2 \end{cases}$$

$$E_1 = A_1 e^{i(\omega t - kx + \phi_1)}$$

$$E_2 = A_2 e^{i(\omega t - ky + \phi_2)}$$

After being reflected at the plane mirrors M_1 and M_2, the two waves are superimposed in the plane of observation B. The amplitudes of the two waves in the plane B is $\sqrt{RT} A_0$ because each wave has been transmitted and reflected once at the beam splitter surface S.



The phase difference ϕ between the two waves is:

$$\phi = \frac{2\pi}{\lambda} 2(SM_1 - SM_2) + \Delta\phi$$

where $\Delta\phi$ accounts for additional phase shifts that may be caused by reflection.

4.3 INTERFEROMETERS

4.3.2 Michelson interferometer

The total complex field amplitude in the plane B is then:

$$E = \sqrt{RT}A_0 e^{i(\omega t+\phi_0)}\left(1 + e^{i\phi}\right)$$

The detector in B cannot follow the rapid oscillations with frequency ω but measures the time-averaged intensity I_T :

$$I_T = \frac{1}{2}c\varepsilon_0 A_0^2 RT(1+e^{i\phi})(1+e^{-i\phi}) = c\varepsilon_0 A_0^2 RT(1+cos\phi) = \frac{1}{2}I_0(1+cos\phi)$$

with $R = T = \frac{1}{2}$ and $I_0 = \frac{1}{2}c\varepsilon_0 A_0^2$

Therefore, I_T can be seen as a function of the variable ϕ. How can I vary ϕ? I need to change the length of one of the interferometer arms.

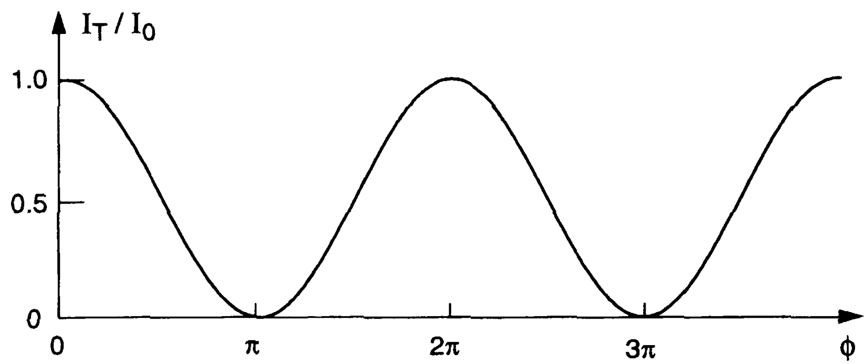
$$\phi = \frac{2\pi}{\lambda}2(SM_1 - SM_2) + \Delta\phi$$

If mirror M_2 s mounted on a carriage) moves along a distance Δy, the optical path difference changes by $\Delta s = 2n\Delta y$ and the phase difference ϕ changes by $2\pi\frac{\Delta s}{\lambda}$.

4.3 INTERFEROMETERS

4.3.2 Michelson interferometer

Figure shows the intensity $I_T(\phi)$ in the plane B as a function of ϕ for a monochromatic incident plane wave.



$$I_T = \frac{1}{2} I_0 (1 + cos\phi)$$

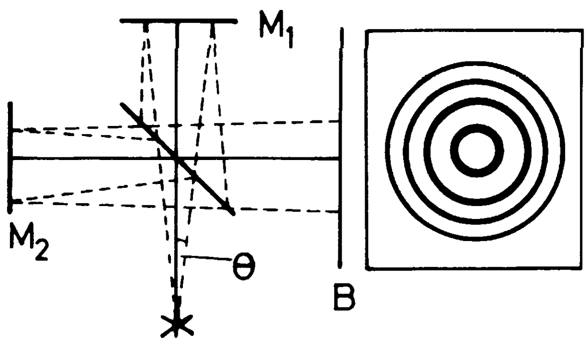
For the maxima at $\phi = 2m\pi\ (m = 0,1,2\ldots)$ the transmitted intensity I_T becomes equal to the incident intensity I_0, which means that the transmission of the interferometer is $T_I = 1$ for $\phi = 2m\pi$.

In the minima for $\phi = (2m + 1)\pi$, the transmitted intensity I_T is zero! The incident plane wave is being reflected back into the source. This illustrates that the MI can be regarded either as a wavelength-dependent filter for the transmitted light, or as a wavelength-selective reflector.

4.3 INTERFEROMETERS

4.3.2 Michelson interferometer

For divergent incident light the path difference between the two waves depends on the inclination angle:



In the plane B an interference pattern of circular fringes, concentric to the symmetry axis of the system, is produced. Moving the mirror M_2 causes the ring diameters to change. The intensity behind a small aperture still follows approximately the function $I_T(\phi)$ in Figure.

The MI can be used for absolute wavelength measurements by counting the number N of maxima in B when the mirror M_2 is moved along a known distance Δy. The wavelength λ is then obtained from:

$$\lambda = \frac{\Delta s}{N} = \frac{2n\Delta y}{N}$$

4.3 INTERFEROMETERS

4.3.2 Michelson interferometer

The MI may be described in another equivalent way, which is quite instructive.

Assume that the mirror M_2 moves with a constant velocity $v = \frac{\Delta y}{\Delta t}$.

A wave with frequency ω and wave vector $\vec{k}$ incident perpendicularly on the moving mirror suffers a Doppler shift:

$$\Delta\omega = \omega - \omega' = 2\vec{k} \cdot \vec{v} = \frac{4\pi}{\lambda} v$$

As a result, the optical path difference is $\Delta s = 2vt = \frac{\lambda t \Delta\omega}{2\pi}$

and the corresponding phase difference $\phi = \frac{2\pi}{\lambda} \Delta s = t\Delta\omega$

Therefore I_T can be seen as:

$$I_T = \frac{1}{2} I_0 (1 + cos\Delta\omega t)$$

$$I_T = \frac{1}{2} I_0 (1 + cos\phi)$$

We recognize as the time-averaged beat signal, obtained from the superposition of two waves with frequencies ω and $\omega' = \omega - \Delta\omega$.

4.3 INTERFEROMETERS

4.3.2 Michelson interferometer

Note that the frequency $\omega = \frac{c}{v}\frac{\Delta\omega}{2}$ f the incoming wave can be measured from the beat frequency $\Delta\omega$, provided the velocity v of the moving mirror is known.

$$\Delta\omega = \frac{4\pi}{\lambda} v$$

The MI with uniformly moving mirror M_2 can be therefore regarded as a device that transforms the high frequency ω $(10^{14} - 10^{15}\ Hz)$ of an optical wave into an easily accessible radiofrequency-range $\frac{v}{c}\omega$.

The maximum path difference Δs that still gives interference fringes in the plane B is limited by the coherence length of the incident radiation.

Using spectral lamps, the coherence length is limited by the Doppler width of the spectral lines and is typically a few centimeters.

With stabilized single-mode lasers, however, coherence lengths of several kilometers can be achieved. In this case, the maximum path difference in the MI is, in general, not restricted by the source but by technical limits imposed by laboratory facilities.

4.3 INTERFEROMETERS

4.3.2 Michelson interferometer

When the incoming radiation is composed of several components with frequencies ω_k, the total amplitude in the plane B of the detector is the sum of all interference amplitudes:

$$E = \sum_k A_k e^{i(\omega_k t+\phi_{0k})}\left(1 + e^{i\phi_k}\right)$$

$$E = \sqrt{RT}A_0 e^{i(\omega t+\phi_0)}\left(1 + e^{i\phi}\right)$$

A detector with a large time constant does not follow the rapid oscillations of the amplitude at frequencies ω_k, but gives a signal proportional to the sum of the intensities I_k. We therefore obtain for the time-dependent total intensity:

$$I_T = \sum_k \frac{1}{2} I_{0k}(1 + cos\phi_k) = \sum_k \frac{1}{2} I_{0k}(1 + cos\Delta\omega_k t)$$

$$I_T = \frac{1}{2} I_0(1 + cos\phi)$$

where the audio frequencies $\Delta\omega_k = \frac{2\omega_k v}{c}$ are determined by the frequencies ω_k of the components and by the velocity v of the moving mirror.

Measurements of these frequencies $\Delta\omega_k$ llows one to reconstruct the spectral components of the incoming wave with frequencies ω_k (**Fourier transform spectroscopy**).

4.3 INTERFEROMETERS

4.3.2 Michelson interferometer

For example, when the incoming wave consists of two components with frequencies ω_1 and ω_2, the interference pattern varies with time according to:

$$\bar{I} = \frac{1}{2} I_{10} \left[1 + cos\left(2\omega_1 \frac{v}{c} t\right)\right] + \frac{1}{2} I_{20} \left[1 + cos\left(2\omega_2 \frac{v}{c} t\right)\right]$$
$$= I_0 \left\{1 + cos\left[(\omega_1 - \omega_2)\frac{v}{c} t\right] cos\left[(\omega_1 + \omega_2)\frac{v}{c} t\right]\right\}$$

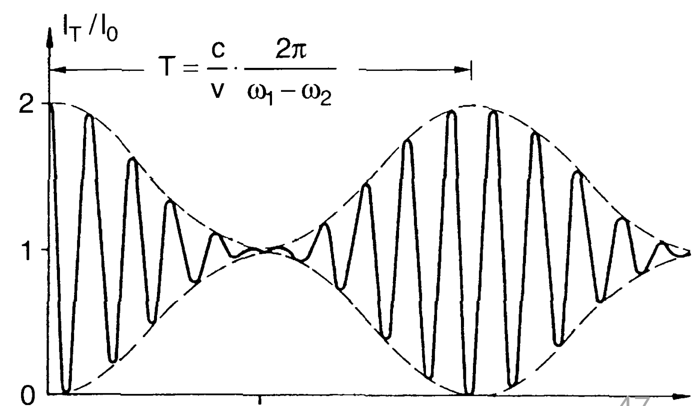
$$\bar{I} = \sum_k \frac{1}{2} I_{0k}(1 + cos\Delta\omega_k t)$$

$$\omega = \frac{c}{v}\frac{\Delta\omega}{2}$$

where we have assumed $I_{10} = I_{20} = I_0$.

and we used the relation $\cos\alpha cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$ with $\alpha = \omega_1 - \omega_2$ e $\beta = \omega_1 + \omega_2$:

This is a **beat signal**, where the amplitude of the interference signal at $(\omega_1 + \omega_2)\frac{v}{c}$ is modulated at the difference frequency $(\omega_1 - \omega_2)\frac{v}{c}$



4.3 INTERFEROMETERS

4.3.2 Michelson interferometer

The **spectral resolution** can roughly be estimated as follows.

If Δy is the path difference traveled by the moving mirror, the number of interference maxima that are counted by the detector is: $N_1 = \frac{2\Delta y}{\lambda_1}$ for an incident wave with the wavelength λ_1, and $N_2 = \frac{2\Delta y}{\lambda_2}$ for an incident wave with wavelength λ_2, with $\lambda_2 < \lambda_1$.

The two wavelengths can be clearly distinguished when $N_2 \geq N_1 + 1$.

Assuming $\lambda_1 = \lambda_2 + \Delta\lambda$, with $\Delta\lambda \ll \lambda$, and $\lambda = \frac{\lambda_1+\lambda_2}{2}$, he spectral resolving power can be expressed as:

$$\frac{\lambda}{\Delta\lambda} = \frac{\frac{1}{2}\left(\frac{2\Delta y}{N_1} + \frac{2\Delta y}{N_2}\right)}{\frac{2\Delta y}{N_1} - \frac{2\Delta y}{N_2}} = \frac{1}{2}\frac{\frac{N_1+N_2}{N_1 N_2}}{\frac{N_2-N_1}{N_1 N_2}} = \frac{1}{2}\frac{N_1+N_2}{N_2-N_1}$$

$$\lambda_1 = \frac{2\Delta y}{N_1}$$

$$\lambda_2 = \frac{2\Delta y}{N_2}$$

4.3 INTERFEROMETERS

4.3.2 Michelson interferometer

Imposing the minimum condition so that the two wavelengths are distinguishable, $N_2 = N_1 + 1$, at the denominator of the relation, we get:

$$\frac{\lambda}{\Delta\lambda} = \frac{1}{2}\frac{N_1 + N_2}{N_2 - N_1}$$

$$\frac{\lambda}{\Delta\lambda} = \frac{N_1 + N_2}{2} = N = \frac{\Delta s}{\lambda}$$

with $N = \frac{N_1+N_2}{2}$.

The spectral resolving power depends only on the count of the interference fringes.

It can be easily verified that the result is identical even in the frequency domain:

$$\frac{\upsilon}{\Delta\upsilon} = N = \frac{\Delta s}{\lambda}$$

4.3 INTERFEROMETERS

4.3.3 Mach-Zender interferometer

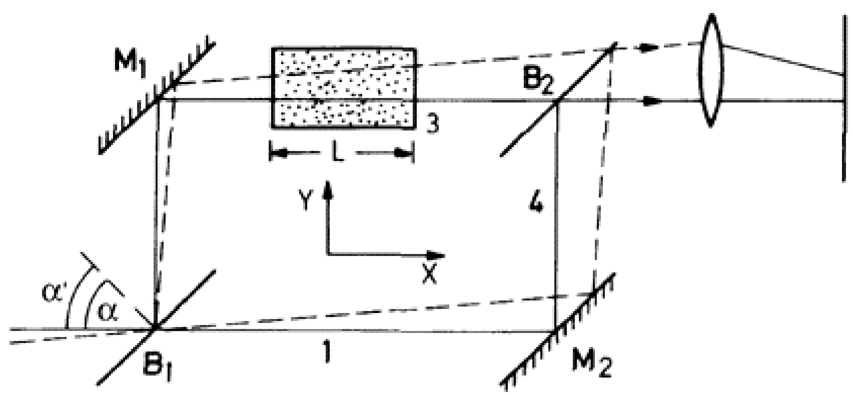
Analogous to the Michelson interferometer, the Mach–Zehnder interferometer is based on the two-beam interference by amplitude splitting of the incoming wave.

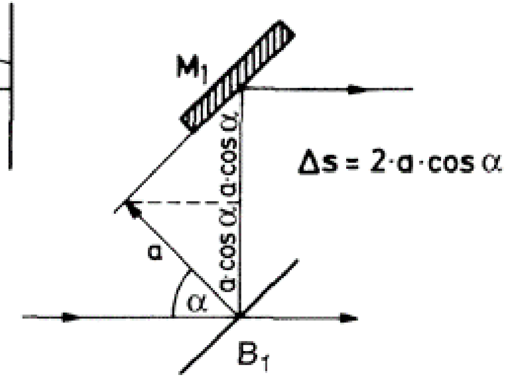
The two waves travel along different paths with a path difference $\Delta s = 2acos\alpha$.

Inserting a transparent object into one arm of the interferometer alters the optical path difference between the two beams.

This results in a change of the interference pattern, which allows a very accurate determination of the refractive index of the sample and its local variation.

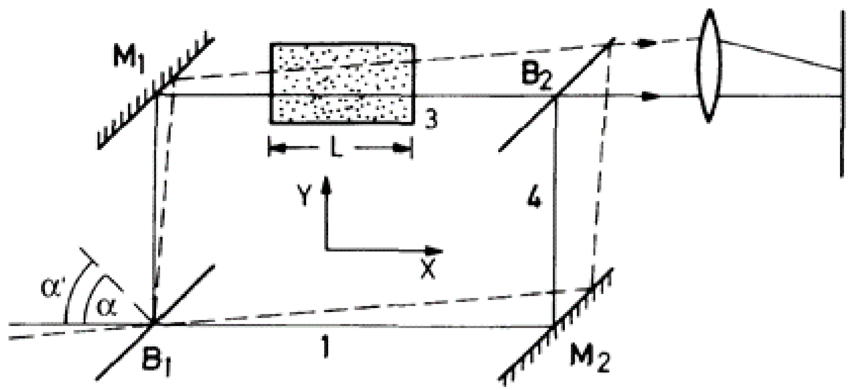
The Mach–Zehnder interferometer may be regarded therefore as a sensitive refractometer.





4.3 INTERFEROMETERS

4.3.3 Mach-Zender interferometer



If the beam splitters B_1 and B_2 and the mirrors M_1 e M_2 are all strictly parallel, the path difference between the two split beams does not depend on the angle of incidence α because the path difference between the beams 1 and 3 is exactly compensated by the same path length of beam 4 between M_2 and B_2.

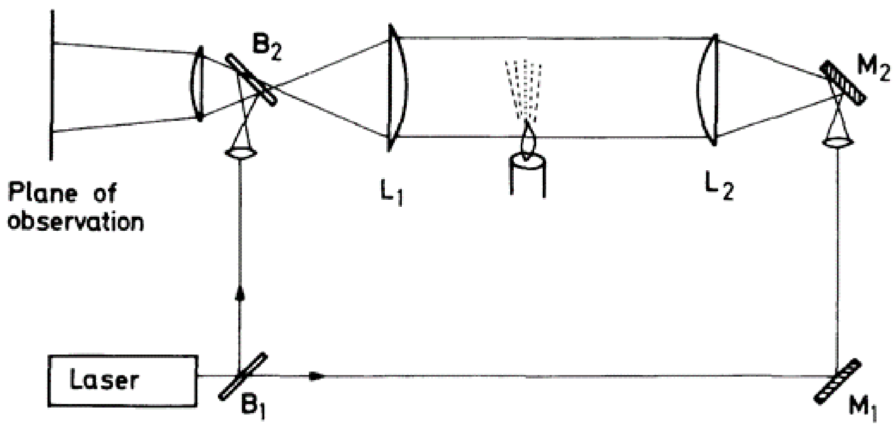
This means that the interfering waves in the symmetric interferometer (without sample) experience the same path difference on the solid path as on the dashed path in Figure.

Without the sample, the total path difference is therefore zero; it is $\Delta s = (n-1)L$ with the sample having the refractive index n in one arm of the interferometer.

4.3 INTERFEROMETERS

4.3.3 Mach-Zender interferometer

Expanding the beam on path 3 gives an extended interference-fringe pattern, which reflects the local variation of the refractive index.



With a beam expander (lenses L_1 and L_2), the laser beam can be expanded up to 10−20 cm and large objects can be tested.

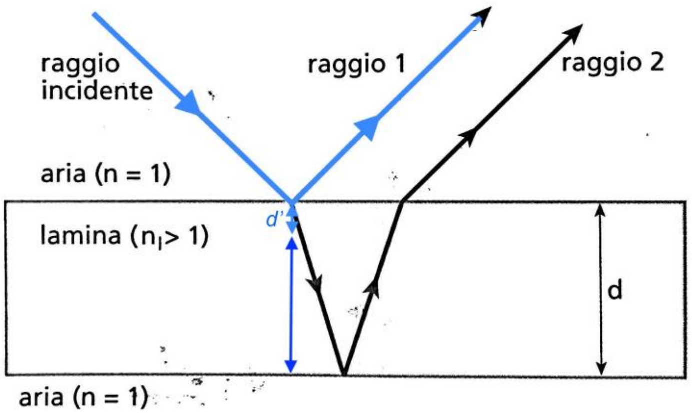
The interference pattern can either be photographed or may be viewed directly with the naked eye or with a television camera. Such a laser interferometer has the advantage that the laser beam diameter can be kept small everywhere in the interferometer, except between the two expanding lenses.

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

In a grating spectrometer, the interfering partial waves emitted from the different grooves of the grating all have the same amplitude.

In contrast, in multiple-beam interferometers these partial waves are produced by multiple reflection at plane or curved surfaces and their amplitude decreases with increasing number of reflections.



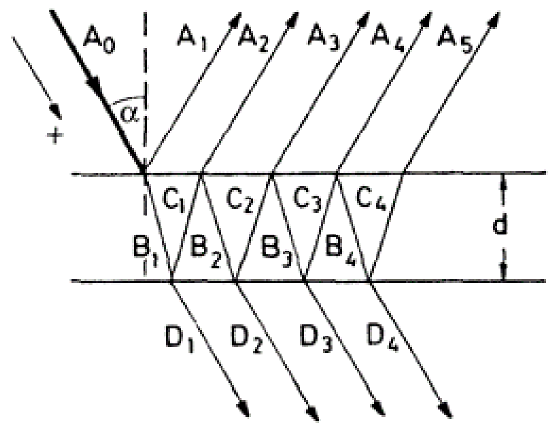
Therefore, the intensity distribution will differ from that found for a grating

$$I_R = I_0 R(\beta)\,\frac{sen^2\left(\frac{N\phi}{2}\right)}{sen^2\left(\frac{\phi}{2}\right)}$$

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

Assume that a plane wave $E = A_0 e^{i(\omega t - kx)}$ is incident at the angle α on a plane transparent plate with two parallel, partially reflecting surfaces.



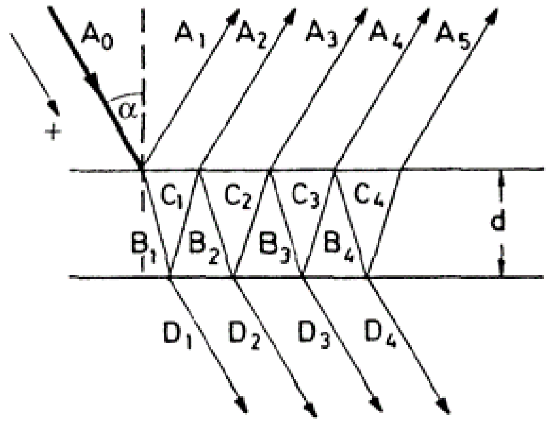
At each surface the amplitude A_i is split into a reflected component $A_R = A_i\sqrt{R}$ and a refracted component $A_T = A_i\sqrt{1-R}$, neglecting absorption.

The reflectivity $R = \frac{I_R}{I_i}$ depends on the angle of incidence α and on the polarization of the incident wave.

Provided the refractive index n is known, R can be calculated from Fresnel's formulas.

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference



From Figure, the following relations are obtained for the amplitudes A_i of waves reflected at the upper surface, the amplitudes B_i of refracted waves, the amplitudes C_i f waves reflected at the lower surface, and the amplitudes D_i of transmitted waves:

$$|A_1| = \sqrt{R}|A_0|$$

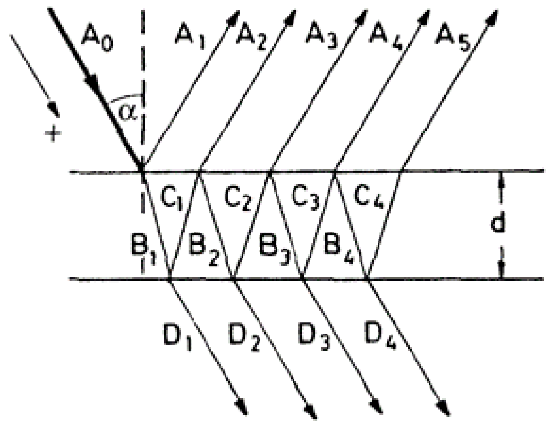
$$|B_1| = \sqrt{1-R}|A_0|$$

$$|C_1| = \sqrt{R}|B_1| = \sqrt{R(1-R)}|A_0|$$

$$|D_1| = \sqrt{1-R}|B_1| = (1-R)|A_0|$$

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference



To follow...

$$|A_2| = \sqrt{1-R}|C_1| = (1-R)\sqrt{R}|A_0|$$

$$|B_2| = \sqrt{R}|C_1| = R\sqrt{(1-R)}|A_0|$$

$$|C_2| = \sqrt{R}|B_2| = R\sqrt{R(1-R)}|A_0|$$

$$|D_2| = \sqrt{1-R}|B_2| = R(1-R)|A_0|$$

and then:

$$|A_3| = \sqrt{1-R}|C_2| = R\sqrt{R}(1-R)|A_0|$$

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

Finally, it is easy to verify that the scheme can be generalized to the equations:

$$|A_{i+1}| = R|A_i| \text{ per } i \geq 2$$

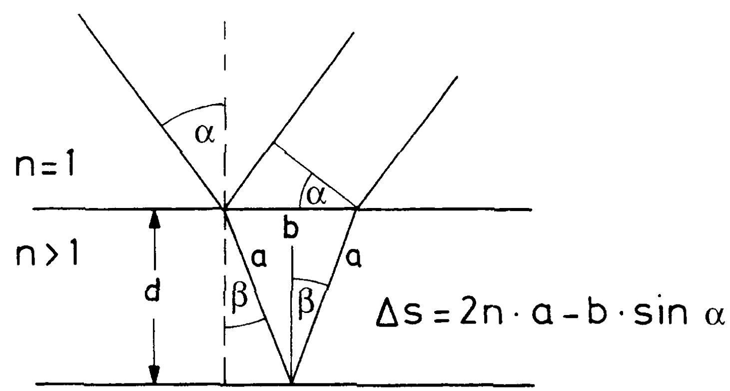
$$|D_{i+1}| = R|D_i| \text{ per } i \geq 1$$

$$|A_1| = \sqrt{R}|A_0|$$
$$|A_2| = (1-R)\sqrt{R}|A_0|$$
$$|A_3| = R\sqrt{R}(1-R)|A_0|$$

$$|D_1| = (1-R)|A_0|$$
$$|D_2| = \sqrt{1-R}|B_2| = R(1-R)|A_0|$$

Two successively reflected partial waves have the optical path difference :

$$\Delta s = 2na - bsen\alpha$$



From Figure, it can be seen that:

$$d = acos\beta \qquad \frac{b}{2} = dtg\beta$$

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

$$\Delta s = 2na - bsen\alpha$$
$$d = acos\beta$$
$$\frac{b}{2} = dtg\beta$$

Replacing:

$$\Delta s = 2n\frac{d}{cos\beta} - 2dtg\beta sen\alpha$$

Because $sen\alpha = nsen\beta$, the optical path difference can be expressed as:

$$\Delta s = 2n\frac{d}{cos\beta} - 2d\frac{sen\beta}{cos\beta}nsen\beta = 2n\frac{d}{cos\beta} - 2nd\frac{1-cos^2\beta}{cos\beta} = 2ndcos\beta$$
$$= 2nd\sqrt{1 - sen^2\beta}$$

This path difference causes a corresponding phase difference:

$$\phi = \frac{2\pi\Delta s}{\lambda} + \Delta\phi$$

where $\Delta\phi$ takes into account possible phase changes caused by the reflections.

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

For instance, the incident wave with amplitude A_1 suffers the phase jump $\Delta\phi = \pi$ while being reflected at the medium with $n > 1$. Including this phase jump, we can write:

$$A_1 = \sqrt{R}|A_0|e^{i\pi} = -\sqrt{R}|A_0|$$

$|A_1| = \sqrt{R}|A_0|$

Taking into account the phase jump, and that all the $|A_i|$ per $i \geq 1$ they do not suffer phase jump because the reflection takes place inside the plate, we will have:

$$|A_1| = -\sqrt{R}|A_0|$$

$$|A_2| = (1-R)\sqrt{R}|A_0|$$

$$|A_{i+1}| = R|A_i| \text{ per } i \geq 2,$$

The total amplitude A of the reflected wave is obtained by summation over all partial amplitudes A_i, taking into account the different phase shifts:

$$A = -\sqrt{R}|A_0| + (1-R)\sqrt{R}|A_0|e^{i\phi} + \sum_{m=3}^{p} A_m e^{i(m-1)\phi}$$

$$= -\sqrt{R}|A_0|\left[1-(1-R)e^{i\phi}\sum_{m=0}^{p-2} R^m e^{im\phi}\right]$$

$|A_1| = \sqrt{R}|A_0|$

$|A_2| = (1-R)\sqrt{R}|A_0|$

$|A_{i+1}| = R|A_i|$ per $i \geq 2$

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

If we suppose an infinite number of reflections, the geometric series $\sum_{m=0}^{p-2}\left(Re^{i\phi}\right)^m$ converges because $|R| < 1$ and then for $p \to \infty$:

$$\sum_{m=0}^{p-2} R^m e^{im\phi} = \frac{1}{1 - Re^{i\phi}}$$

Replacing:

$$A = -\sqrt{R}|A_0|\left[1 - (1-R)e^{i\phi}\sum_{m=0}^{p-2} R^m e^{im\phi}\right]$$

$$A = -\sqrt{R}|A_0|\left[1 - \frac{(1-R)e^{i\phi}}{1 - Re^{i\phi}}\right] = \sqrt{R}|A_0|\frac{1 - e^{i\phi}}{1 - Re^{i\phi}}$$

The intensity of the reflected wave is:

$$I_R = 2c\varepsilon_0 AA^*$$

and then using the previous expression:

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

$$A = \sqrt{R}|A_0|\frac{1-e^{i\phi}}{1-Re^{i\phi}}$$

$$I_R = 2c\varepsilon_0 AA^* = 2c\varepsilon_0 R|A_0|^2 \frac{1-e^{i\phi}}{1-Re^{i\phi}} \cdot \frac{1-e^{-i\phi}}{1-Re^{-i\phi}}$$

$$= 2c\varepsilon_0 R|A_0|^2 \frac{2-e^{-i\phi}-e^{i\phi}}{1-Re^{-i\phi}-Re^{i\phi}+R^2} = 2c\varepsilon_0 R|A_0|^2 \frac{2-2cos\phi}{1+R^2-2Rcos\phi}$$

With $I_0 = 2c\varepsilon_0|A_0|^2$ and using the trigonometric relation: $sen^2\left(\frac{\phi}{2}\right) = \frac{1-cos\phi}{2}$, the intensity of the reflected wave can be written as:

$$I_R = I_0 R \frac{4sen^2\left(\frac{\phi}{2}\right)}{1+R^2-2R+2R-2Rcos\phi} = I_0 R \frac{4sen^2\left(\frac{\phi}{2}\right)}{(1-R)^2+4Rsen^2\left(\frac{\phi}{2}\right)}$$

In an analogous way, we find for the total transmitted amplitude. Being $|D_{i+1}| = R|D_i|$ for $i \geq 1$, together with $|D_1| = (1-R)|A_0|$ and $|D_2| = R(1-R)|A_0|$:

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

$$D = \sum_{m=1}^{\infty} D_m e^{i(m-1)\phi} = (1-R)A_0 \sum_{m=0}^{\infty} R^m e^{im\phi}$$

$|D_1| = (1-R)|A_0|$

$|D_2| = \sqrt{1-R}|B_2| = R(1-R)|A_0|$

$|D_{i+1}| = R|D_i|$ per $i \geq 1$

The geometric series converges:

$$D = \frac{(1-R)A_0}{1-Re^{i\phi}}$$

The intensity of the transmitted wave will be (the calculations to the denominator are identical to the case of the reflected wave):

$$I_T = 2c\varepsilon_0 DD^* = 2c\varepsilon_0 \frac{(1-R)^2|A_0|^2}{(1-R)^2 + 4Rsen^2\left(\frac{\phi}{2}\right)}$$

$$= I_0 \frac{(1-R)^2}{(1-R)^2 + 4Rsen^2\left(\frac{\phi}{2}\right)}$$

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

Eqiuations I_R and I_T are called the **Airy formulas**.

Using the abbreviation:

$$F = \frac{4R}{(1-R)^2}$$

Airy equations can be written in the form:

$$I_R = I_0 \frac{F sen^2\left(\frac{\phi}{2}\right)}{1 + F sen^2\left(\frac{\phi}{2}\right)}$$

$$I_T = I_0 \frac{1}{1 + F sen^2\left(\frac{\phi}{2}\right)}$$

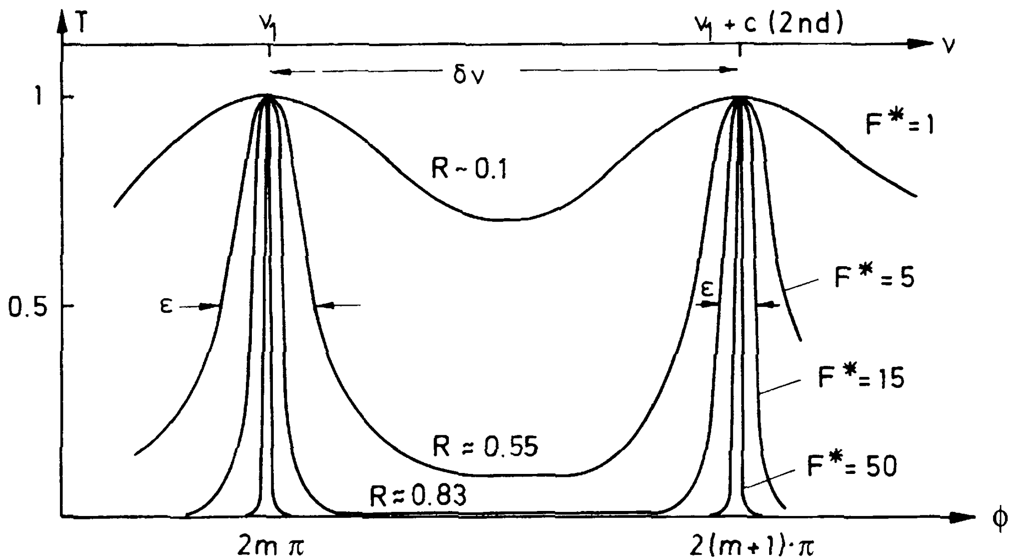
$$I_R = I_0 R \frac{4 sen^2\left(\frac{\phi}{2}\right)}{(1-R)^2 + 4R sen^2\left(\frac{\phi}{2}\right)}$$

$$I_T = I_0 \frac{(1-R)^2}{(1-R)^2 + 4R sen^2\left(\frac{\phi}{2}\right)}$$

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

Figure illustrates I_T for different values of the reflectivity R.



$$I_T = I_0 \frac{1}{1 + F sen^2\left(\frac{\phi}{2}\right)}$$

$$F = \frac{4R}{(1-R)^2}$$

The maximum transmittance ($T = I_T/I_0$) is $T = 1$ for $\phi = 2m\pi$. At these maxima $I_T = I_0$, therefore the reflected intensity $I_R = 0$.

The minima transmittance are when $sen^2\left(\frac{\phi}{2}\right) = 1$, corresponding to:

$$T_{min} = \frac{1}{1+F} = \frac{1}{1 + \frac{4R}{(1-R)^2}} = \frac{(1-R)^2}{(1-R)^2 + 4R} = \left(\frac{1-R}{1+R}\right)^2$$

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

Free spectral range and Finesse

The frequency range $\delta\upsilon$ between two maxima is the free spectral range of the interferometer.

With $\phi = \frac{2\pi\Delta s}{\lambda}$ and using

$$\Delta s = 2ndcos\beta = 2nd\sqrt{1 - sen^2\beta} = 2nd\sqrt{1 - \frac{sen^2\alpha}{n^2}} = 2d\sqrt{n^2 - sen^2\alpha}$$

(where Snell's law $sen\alpha = nsen\beta$ was used), the FSR will be:

$$\delta\upsilon = \frac{c}{\Delta s} = \frac{c}{2d\sqrt{n^2 - sen^2\alpha}}$$

For vertical incidence ($\alpha = 0$), the free spectral range becomes:

$$\delta\upsilon = \frac{c}{2nd}$$

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

Free spectral range and Finesse

The full-width half-maximum $\epsilon = |\phi_1 - \phi_2|$ with $I(\phi_1) = I(\phi_2) = I_0/2$ of the transmission maxima can be calculated using the expression previously derived for I_T:

$$\frac{I_0}{2} = I_0 \frac{1}{1 + F sen^2\left(\frac{\phi_1}{2}\right)}$$

$$sen^2\left(\frac{\phi_1}{2}\right) = \frac{1}{F}$$

$$\phi_1 = 2arcsen\frac{1}{\sqrt{F}} = 2arcsen\left(\frac{1-R}{2\sqrt{R}}\right)$$

and so:

$$\epsilon = 4arcsen\left(\frac{1-R}{2\sqrt{R}}\right)$$

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

Free spectral range and Finesse

Assuming $R \approx 1$, then $(1 - R) \ll R$ and so:

$$\epsilon = 4arcsen\left(\frac{1-R}{2\sqrt{R}}\right)$$

$$F = \frac{4R}{(1-R)^2}$$

$$\epsilon \approx 4\left(\frac{1-R}{2\sqrt{R}}\right) = 2\left(\frac{1-R}{\sqrt{R}}\right) = \frac{4}{\sqrt{F}}$$

Now we need to convert $\epsilon = \Delta\phi$ in frequency units .

Starting from the definition of $\phi = \frac{2\pi}{\lambda}\Delta s$ and combining it with expressions: $\lambda = \frac{c}{\nu}$ and $\delta\upsilon = \frac{c}{\Delta s}$, one obtain $\phi = \frac{2\pi\upsilon}{\delta\upsilon}$, and $\epsilon = \Delta\phi = \frac{2\pi\Delta\upsilon}{\delta\upsilon}$, from which:

$$\Delta\upsilon = \frac{\epsilon}{2\pi}\delta\upsilon = \frac{c}{2nd}\frac{1-R}{\pi\sqrt{R}}$$

$$\delta\upsilon = \frac{c}{2nd}$$

The ratio $\delta\upsilon/\Delta\upsilon$ of free spectral range to the halfwidth of the transmission maxima is called the **finesse F^*** of the interferometer:

$$F^* = \frac{\delta\upsilon}{\Delta\upsilon} = \frac{\pi\sqrt{R}}{1-R} = \frac{\pi}{2}\sqrt{F}$$

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

Free spectral range and Finesse

$$F^* = \frac{\pi\sqrt{R}}{1-R} = \frac{\pi}{2}\sqrt{F}$$

Since we have assumed an ideal plane-parallel plate with a perfect surface quality, the finesse is determined only by the reflectivity R of the surfaces. In practice, however, deviations of the surfaces from an ideal plane and slight inclinations of the two surfaces cause imperfect superposition of the interfering waves. This results in a decrease and a broadening of the transmission maxima, which decreases the total finesse.

Spectral resolution

The spectral resolution, $\upsilon/\Delta\upsilon$ of an interferometer is determined by the free spectral range $\delta\upsilon$ and by the finesse F^*.

$$F^* = \frac{\delta\upsilon}{\Delta\upsilon}$$

Two incident waves with frequencies υ_1 and $\upsilon_2 = \upsilon_1 + \Delta\upsilon$ can still be resolved if their frequency separation $\Delta\upsilon$ is larger than $\delta\upsilon/F^*$, which means that their peak separation should be larger than their full halfwidth.

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

Spectral resolution

So the spectral resolving power of an interferometer os:

$$\frac{\upsilon}{\Delta\upsilon} = \frac{\upsilon}{\delta\upsilon} F^*$$

This can be also expressed by the optical path differences Δs between two successive partial waves, using $\delta\upsilon = \frac{c}{\Delta s}$:

$$\frac{\upsilon}{\Delta\upsilon} = \frac{\upsilon}{c} \Delta s F^* = F^* \frac{\Delta s}{\lambda}$$

The resolving power of an interferometer is the product of finesse F^* and optical path difference $\frac{\Delta s}{\lambda}$ in units of the wavelength. A comparison with the resolving power of a grating spectrometer with N grooves, $\frac{\upsilon}{\Delta\upsilon} = mN = N\frac{\Delta s}{\lambda}$, shows that the finesse F^* can indeed be regarded as the effective number of interfering partial waves.

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

If we consider absorption losses for each reflective surface, the relation between the reflectivity and the transmittivity must take into account a contribution A due to losses, namely $A = 1 - R - T$. As a consequence, the total amplitude of the transmitted wave must be modified:

$$D = \frac{(1-R-A)A_0}{1-Re^{i\phi}}$$

$$D = \frac{(1-R)A_0}{1-Re^{i\phi}}$$

and the intensity of the transmitted wave is (the denominator calculations are identical to the case of the reflected wave):

$$I_T = 2c\varepsilon_0 DD^* = I_0 \frac{(1-R-A)^2}{(1-R)^2 + 4Rsen^2\left(\frac{\phi}{2}\right)}$$

Introducing the same factor $F = \frac{4R}{(1-R)^2}$, the total intensity of the transmitted wave when losses are included is:

$$I_T = I_0 \frac{(1-R-A)^2}{(1-R)^2} \frac{1}{1+4Rsen^2\left(\frac{\phi}{2}\right)} = I_0 \frac{T^2}{(A+T)^2} \frac{1}{1+4Rsen^2\left(\frac{\phi}{2}\right)}$$

4.3 INTERFEROMETERS

4.3.4 Multiple-Beam Interference

$$I_T = I_0 \frac{T^2}{(A+T)^2} \frac{1}{1 + 4R sen^2\left(\frac{\phi}{2}\right)}$$

The absorption causes two effects:

1. The maximum transmittance is decreased by the fact

$$\frac{I_T}{I_0} = \frac{T^2}{(A+T)^2} = \frac{T^2}{(1-R)^2} < 1$$

Note that even a small absorption of each reflecting surface results in a drastic reduction of the total transmittance. For $R = 0.9$, $T = 0.05$ and $A = 0.05$, the factor is $\frac{T^2}{(1-R)^2} = 0.25.$

2. The factor F becomes:

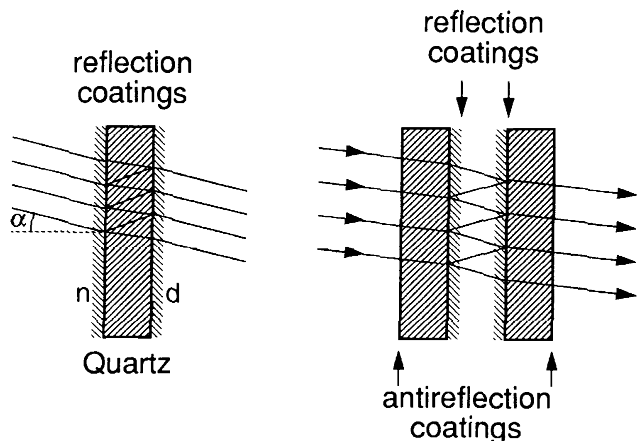
$$F = \frac{4R}{(1-R)^2} = \frac{4(1-T-A)}{(T+A)^2}$$

which decreases with increasing A. Since $F^* = \frac{\pi}{2}\sqrt{F}$, this makes the transmission peaks broader because of the decreasing number of interfering partial waves.

4.3 INTERFEROMETERS

4.3.5 Plane Fabry-Perot Interferometer

A practical realization of the multiple beam-interference discussed in this section may use either a solid plane-parallel glass or fused quartz plate with two coated reflecting surfaces (Fabry–Perot etalon) or two separate plates, where one surface of each plate is coated with a reflection layer.



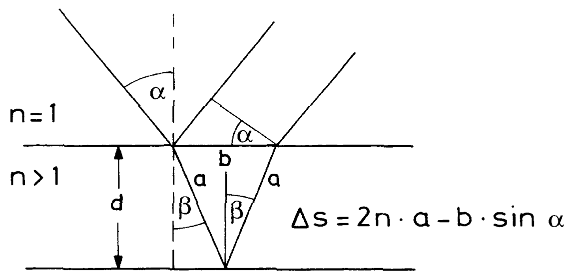
The two reflecting surfaces are opposed and are aligned to be as parallel as achievable (Fabry–Perot interferometer). The outer surfaces are coated with antireflection layers in order to avoid reflections from these surfaces that might overlap the interference pattern.

Both devices can be used for parallel as well as for divergent incident light. We now discuss only the case of illumination with parallel light.

4.3 INTERFEROMETERS

4.3.5 Plane Fabry-Perot Interferometer

In laser spectroscopy, etalons are mainly used as wavelength-selective transmission filters within the laser resonator to narrow the laser bandwidth. The wavelength λ_m or frequency υ_m for the transmission maximum of m-th order, where the optical path between successive beams is $\Delta s = m\lambda$, will be:



$$\lambda_m = \frac{2nd}{m} cos\beta \qquad \upsilon_m = \frac{mc}{2ndcos\beta}$$

For all wavelengths $\lambda = \lambda_m$ with $m = 0,1,2 \ldots$ in the incident light, the phase difference between the transmitted partial waves becomes $\delta = 2m\pi$ and the transmitted intensity is:

$$I_T = I_0 \frac{T^2}{(A+T)^2} \frac{1}{1 + 4Rsen^2(m\pi)} = I_0 \frac{T^2}{(A+T)^2} = I_0 \frac{T^2}{(1-R)^2}$$

The reflected waves interfere destructively for $\lambda = \lambda_m$ and the reflected intensity will be zero. Note, however, that this is only true for $A \ll 1$ and infinitely extended plane waves, where the different reflected partial waves completely overlap.

4.3 INTERFEROMETERS

4.3.6 Multilayer Dielectric Coatings

The constructive interference found for the reflection of light from plane-parallel interfaces between two regions with different refractive indices can be utilized to produce highly reflecting, essentially absorption-free mirrors..

The reflectivity R of a plane interface between two regions with complex refractive indices $n_1 = n_1' - i\kappa_1$ and $n_2 = n_2' - i\kappa_2$ can be calculated from Fresnel's formulas. It depends on the angle of incidence α and on the direction of polarization.

For vertical incidence ($\alpha = 0$), one obtains from Fresnel's formulas, for incident light polarized parallel and perpendicular to the plane of incidence, the reflectivity:

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

To achieve maximum reflectivities, the numerator $(n_1 - n_2)^2$ should be maximized and the denominator minimized. Since n_1 is always larger than one, this implies that n_2 should be as large as possible.

4.3 INTERFEROMETERS

4.3.6 Multilayer Dielectric Coatings

Unfortunately, the Kramer-Kroning dispersion relations :

$$\alpha = \frac{Ne^2}{4\varepsilon_0 mc} \frac{\left(\frac{\gamma}{2}\right)}{(\omega_0 - \omega)^2 + \left(\frac{\gamma}{2}\right)^2}$$

$$n' = 1 + \frac{Ne^2}{4\varepsilon_0 m\omega_0} \frac{\omega_0 - \omega}{(\omega_0 - \omega)^2 + \left(\frac{\gamma}{2}\right)^2}$$

imply that a large value of n also causes large absorption.

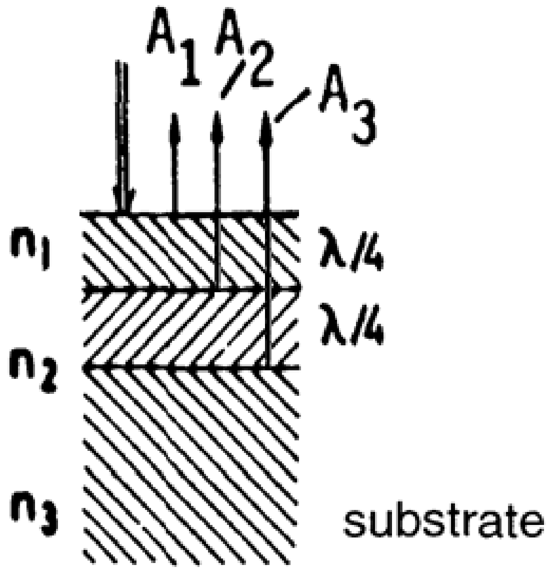
The situation can be improved by selecting reflecting materials with low absorption (which then necessarily also have low reflectivity) but using many layers with alternating high and low refractive index n.

Choosing the proper optical thickness nd of each layer allows constructive interference between the different reflected amplitudes to be achieved. Reflectivities of up to $R = 0.9995$ have been reached.

4.3 INTERFEROMETERS

4.3.6 Multilayer Dielectric Coatings

Figure illustrates such constructive interference for the example of a two-layer coating..



The layers with refractive indices n_1 and n_2 and thicknesses d_1 and d_2 are evaporated onto an optically smooth substrate with the refractive index n_3.

The phase differences between all reflected components have to be $\delta_m = 2m\pi$, with $m = 1,2,3 \ldots$, for constructive interference.

4.3 INTERFEROMETERS

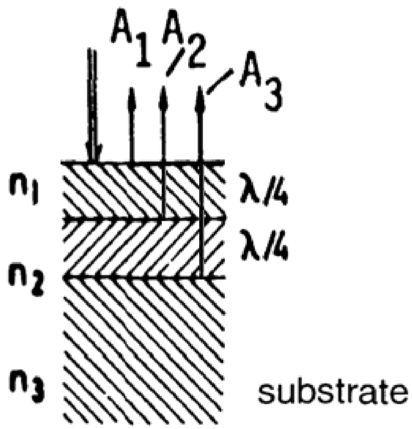
4.3.6 Multilayer Dielectric Coatings

Assuming $n_1 > n_2$ and $n_3 > n_2$, and taking into account the phase shift π at reflection from an interface with a larger refractive index than that of the foregoing layer, we obtain the conditions between $\overrightarrow{A_1}$ and $\overrightarrow{A_2}$:

$$\delta_{(m=1)} = \frac{2\pi}{\lambda}\Delta s + \pi = \frac{2\pi}{\lambda} 2n_1 d_1 + \pi = 2\pi$$

and

$$n_1 d_1 = \frac{\lambda}{4}$$



Similarly, the condition between $\overrightarrow{A_2}$ and $\overrightarrow{A_3}$ for the layer d_2 is:

$$\delta_{(m=1)} = \frac{2\pi}{\lambda}\Delta s + \pi = \frac{2\pi}{\lambda}(2n_2 d_2) + \pi = 2\pi$$

from which:

$$n_2 d_2 = \frac{\lambda}{4}$$

4.3 INTERFEROMETERS

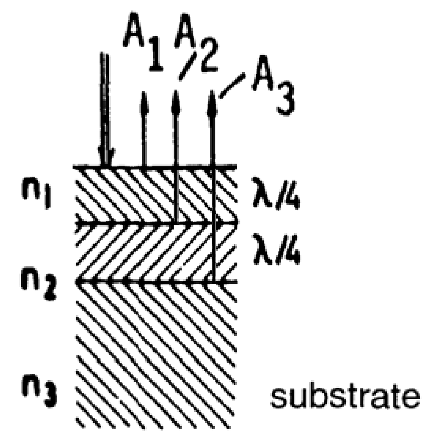
4.3.6 Multilayer Dielectric Coatings

The reflected amplitudes can be calculated from Fresnel's formulas. The total reflected intensity is obtained by summation over all reflected amplitudes taking into account the correct phase. The refractive indices are now selected such that $\sum_i A_i$ becomes a maximum. The calculation is still feasible for our example of a two-layer coating and yields for the three reflected amplitudes:

$$A_1 = \sqrt{R_1} A_0$$

$$A_2 = \sqrt{R_2}(1 - \sqrt{R_1}) A_0$$

$$A_3 = \sqrt{R_3}(1 - \sqrt{R_2})(1 - \sqrt{R_1}) A_0$$



with reflectivities R_i given by the Fresnel formula.

$$R_i = \left(\frac{n_i - n_{i+1}}{n_i + n_{i+1}} \right)^2$$

For example, if we consider:

$$|n_1| = 1.6$$

$$|n_2| = 1.2$$

$$|n_3| = 1.45$$

4.3 INTERFEROMETERS

4.3.6 Multilayer Dielectric Coatings

you get:

$$A_1 = 0.231 A_0$$

$$A_2 = 0.143 A_0$$

$$A_3 = 0.094 A_0$$

which leads to the total amplitude:

$$A_R = \sum_i A_i = 0.468 A_0$$

corresponding to a reflected intensity of:

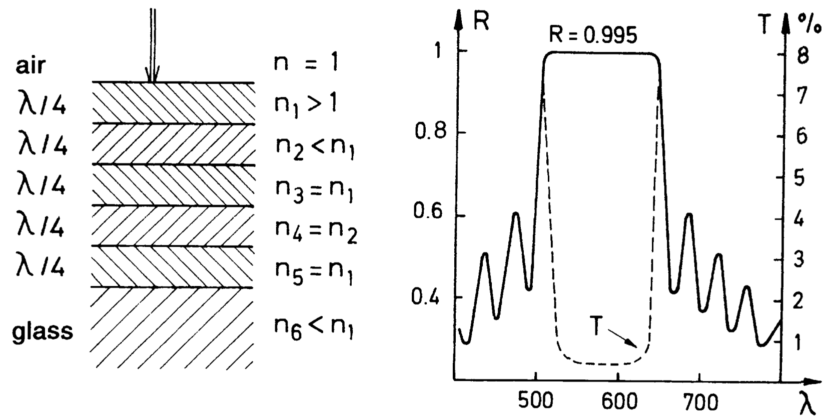
$$I_R = 0.22 I_0$$

This example illustrates that for materials with low absorption, many layers are necessary to achieve a high reflectivity.

4.3 INTERFEROMETERS

4.3.6 Multilayer Dielectric Coatings

Figure depicts schematically the composition of a dielectric multilayer mirror.



he calculation and optimization of multilayer coatings with up to 20 layers becomes very tedious and time consuming and is therefore performed using computer programs. By proper selection of different layers with slightly different optical path lengths, one can achieve a high reflectivity over an extended spectral range. Currently, “broad-band” reflectors are available with reflectivity of $R > 0.99$ within the spectral range $\lambda_0 \pm 0.2\lambda_0$, while the absorption losses are less than 0.2%.

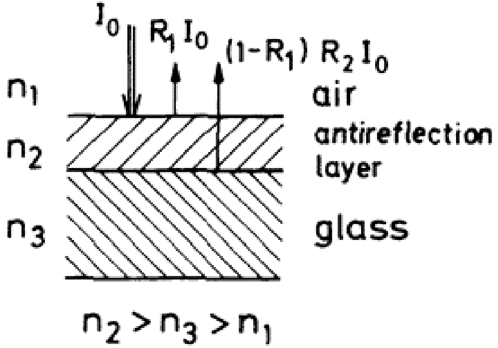
At such low absorption losses, the scattering of light from imperfect mirror surfaces may become the major loss contribution. In this case, the mirror substrate must be of high quality, with roughness imperfections smaller than $\frac{\lambda}{20}$.

4.3 INTERFEROMETERS

4.3.6 Multilayer Dielectric Coatings

Instead of maximizing the reflectivity of a dielectric multilayer coating through constructive interference, it is, of course, also possible to minimize it by destructive interference. Such antireflection coatings are commonly used to minimize unwanted reflections from the many surfaces of multiple-lens camera objectives, which would otherwise produce an annoying background illumination of the photomaterial. In laser spectroscopy such coatings are important for minimizing reflection losses of optical components inside the laser resonator and for avoiding reflections from the back surface of output mirrors, which would introduce undesirable couplings, thereby causing frequency instabilities of single-mode lasers.

Using a single layer, the reflectivity reaches a minimum only for a selected wavelength. We obtain $I_R = 0$ for $\delta = (2m+1)\pi$, if the two amplitudes $A_1 = \sqrt{R_1}A_0$ and $A_2 = \sqrt{R_2}(1-\sqrt{R_1})A_0$ reflected by the interfaces (n_1, n_2) and (n_2, n_3) are equal.



With multilayer antireflection coatings the reflectivity can be decreased below 0.2% for an extended spectral range.

EXERCISE 1

EXERCISE 1

A spectrometer with groove size $\mathrm{d} = 0.56\,\mu m$ shall be used in first order for a wavelength range around $500\,nm$. What is the optimum blaze angle, if the geometry of the spectrometer allows an angle of incidence α about 20°?

Let's start from the relation between the blaze angle θ with the angle of incidence α and the diffraction angle β:

$$\theta = \frac{\alpha - \beta}{2}$$

The diffraction angle β can be determined by the grating equation with $m = 1$:

$$d(sen\alpha + sen\beta) = \lambda$$

EXERCISE 1

thus:

$$\beta = arcsen\left(sen\alpha - \frac{\lambda}{d}\right) = arcsen\left(0.342 - \frac{0.5\ \mu m}{0.56\ \mu m}\right)$$
$$= arcsen\left(0.342 - \frac{0.5\ \mu m}{0.56\ \mu m}\right) = -33.4°$$

Then the blaze angle θ is:

$$\theta = \frac{\alpha - \beta}{2} = \frac{20° + 33.4°}{2} = 26.7°$$

EXERCISE 2

EXERCISE 2

A fluorescence spectrum shall be measured with a spectral resolution of $10^{-2}\ nm$ at $\lambda = 500\ nm$. The experimentor decides to use a crossed arrangement of a grating spectrometer (linear dispersion: 0.5 nm/mm) and a Fabry–Perot interferometer with $R = 0.97$. Estimate the optimum combination of spectrometer slit width and Fabry–Perot interferometer plate separation.

The spectral resolution required to measure the fluorescence spectrum is:

$$R = \frac{\lambda}{\Delta\lambda} \geq \frac{500\ nm}{10^{-2}\ nm} = 5 \cdot 10^{4}$$

We determine the separation between the plates of the Fabry-Perot interferometer in order to reach the required spectral resolving power:

$$\frac{\upsilon}{\Delta\upsilon} = \frac{\upsilon}{c}\Delta s F^{*} = F^{*}\frac{\Delta s}{\lambda}$$

EXERCISE 2

$$\frac{\upsilon}{\Delta\upsilon} = \frac{\lambda}{\Delta\lambda} = F^* \frac{\Delta s}{\lambda}$$

where F^* is the finesse and Δs is the optical path difference between two consecutively reflected partial waves .

The plate separation d of the Fabry–Perot interferometer has to be:

$$\Delta s = 2d$$

The finesse can be determined:

$$F^* = \frac{\pi\sqrt{R}}{1 - R} = \frac{3.14 \cdot 0.985}{0.03} = 103$$

Then the separation between the two plates must be:

$$d = \left(\frac{\lambda}{\Delta\lambda}\right)\frac{\lambda}{2F^*} = 5 \cdot 10^4 \frac{0.5\ \mu m}{2 \cdot 103} = 121\ \mu m$$

EXERCISE 2

When the interferometer is combined with the grating spectrometer, the spectral interval $\Delta\lambda$ transmitted by the spectrograph should be smaller than the Free Spectral Range $\delta\upsilon$, in order to avoid the overlap of different orders.

The Free Spectral Range ($\delta\upsilon$) of the interferometer is equal to:

$$\delta\upsilon = \frac{c}{2d}$$

Using the relation (understood as absolute values):

$$\delta\lambda = \frac{c}{\upsilon^2}\delta\upsilon$$

one obtain:

$$\delta\lambda = \frac{c}{c^2}\lambda^2\frac{c}{2d} = \frac{\lambda^2}{2d} = \frac{250 \cdot 10^3\ nm^2}{2 \cdot 121 \cdot 10^3\ nm} = 1\ nm$$

EXERCISE 2

Then the spectral resolution of the gratin spectrometer must be:

$$\Delta\lambda = \frac{\partial\lambda}{\partial x}\Delta s \leq 1\,nm$$

with Δs equal to the width of the output slit of the spectrometer.

Using the linear dispersion of the grating, we determine:

$$\Delta s \leq 1\,nm\left(\frac{\partial\lambda}{\partial x}\right)^{-1} = \frac{1\,nm}{5\cdot 10^{-1}\,\frac{nm}{mm}} = 2\,mm$$