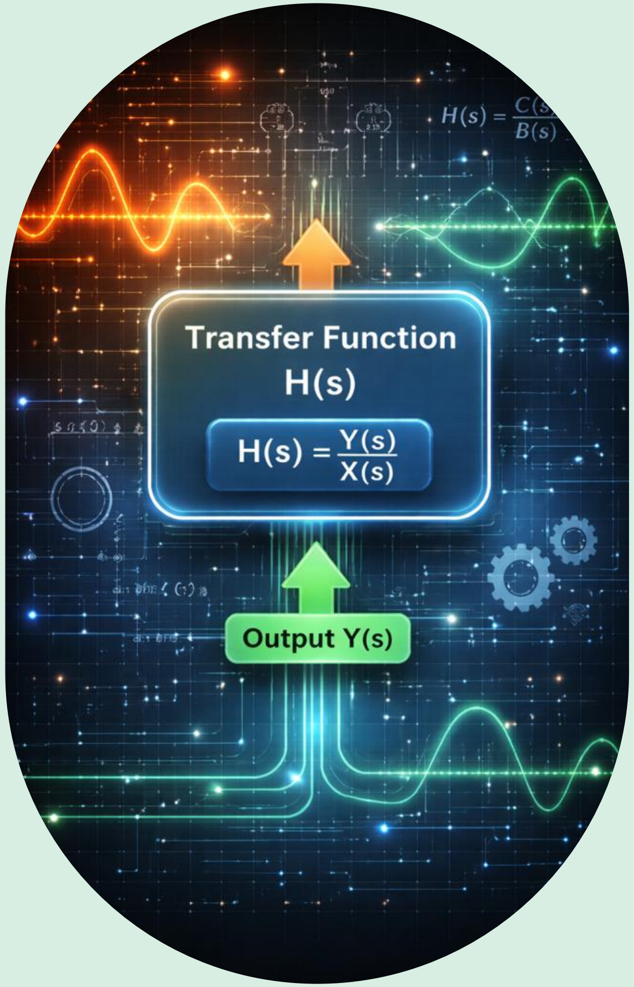


Chapter 2

Transfer functions

2.1 Mathematical models

Most of stimuli provided to a sensors are not electrical. Therefore, from its input to the output a sensor may perform several signal conversion steps before it produces and outputs an electrical signal.

For example, in fiber optic pressure sensor, the applied pressure results in strain in the fiber which in turn causes deflection in its refractive index, which in turn changes the optical transmission and modulates the photon density and finally the photon flux is detected by a photodiode and converted into electric current.

How can we model the input-output relation? How can we employ this relation to determine an unknown input stimulus from the sensor's electric output?

2.1 Mathematical models

An ideal or theoretical **input–output (stimulus–response) relationship** exists for every sensor. For an ideal sensor, the output would always represent the true value of the stimulus. This ideal input–output relationship may be expressed in the form of a table of values, graph, mathematical formula, or as a solution of a mathematical equation.

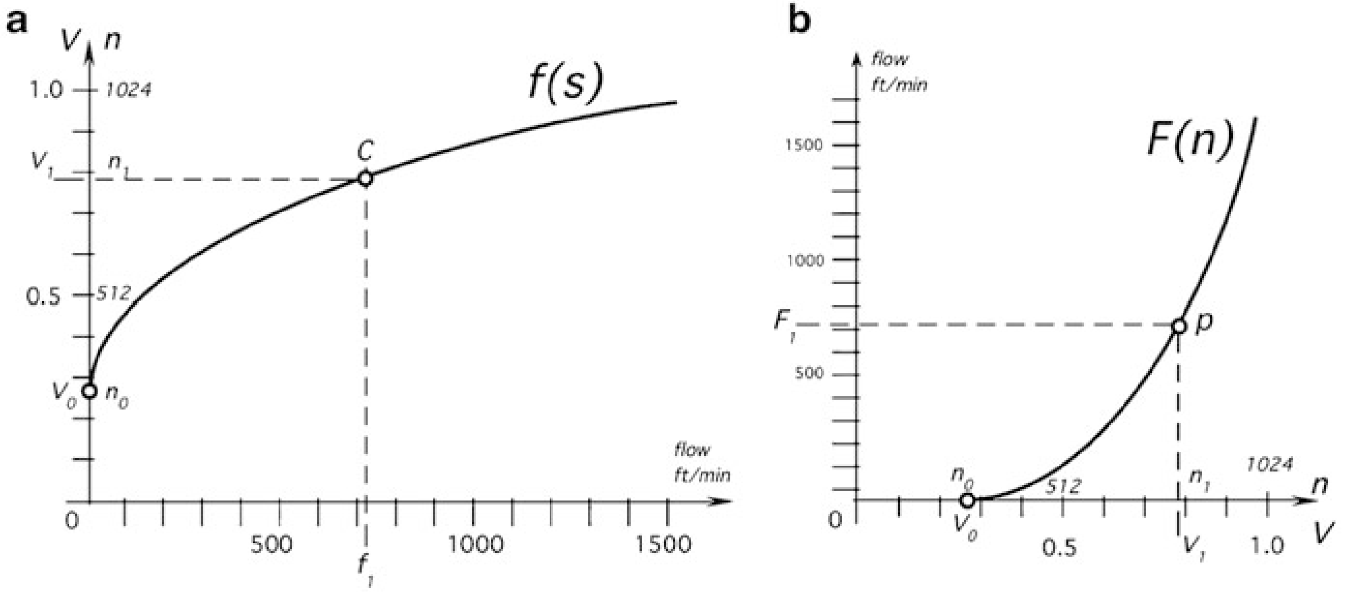
If the input–output function is time invariant, it is called **transfer function**.

A transfer function represents a relation between the input stimulus s and the electrical signal E produced by the sensor at its output. This relation can be written as $\boldsymbol{E}=\boldsymbol{f}(\boldsymbol{s})$.

Normally, the stimulus is unknown while the output signal is measured and thus becomes known. The value of E that becomes known during measurement is a number (voltage, current, digital count, etc.) that represents stimulus s.

2.1 Mathematical models

In real life, any sensor is related to a measuring system. One of the functions of the system is to “break the code E” and infer the unknown value of s from the measured value of E. Thus, the measurement system shall employ an inverse transfer function $\boldsymbol{s} = \boldsymbol{f}^{-1}(\boldsymbol{E}) = \boldsymbol{F}(\boldsymbol{E})$, to compute value of the stimulus s. It is usually desirable to determine a transfer function not just of a sensor alone, but rather of a system comprising the sensor and its interface circuit.



2.1 Mathematical models

Preferably, a physical or chemical law that forms a basis for the sensor's operation should be known. If such a law can be expressed in form of a mathematical formula, often it can be used for calculating the sensor's inverse transfer function by inverting the formula and computing the unknown value of s from the measured output E.

For example, a linear resistive potentiometer is used for sensing displacement d, representing the stimulus s. The Ohm's law can be applied for computing the transfer function. The electric output E is the measured voltage V while the inverse transfer function is given as:

$$d = F(E) = \frac{D}{V_0} \cdot V$$

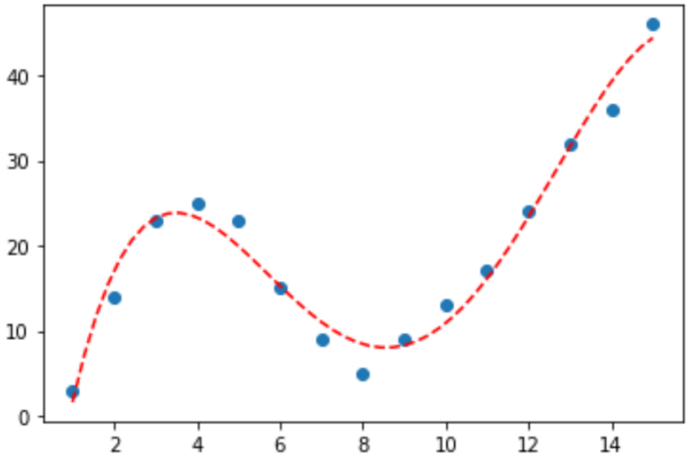
where V_0 is the reference voltage and D is the maximum displacement.

2.1 Mathematical models

In practice, readily solvable formulas for many transfer functions, especially for complex sensors, does not exist and various approximations of the direct and inverse transfer functions are used.

Approximation is a selection of a suitable mathematical expression that can fit the experimental data as close as possible. The act of approximation can be seen as a curve fitting of the experimentally observed values into the approximating function.

The approximating function should be simple enough for ease of computation and inversion and other mathematical treatments, for example, for computing a derivative to find the sensor's sensitivity. The selection of such a function requires some mathematical experience.



2.1 Mathematical models

The simplest model of a transfer function is linear. It is described by the following equation:

$$E = A + Bs$$

The intercept **A** is the output signal E at zero input signal $s = 0$ (**intercept**). The slope of the line is **B**, and it is also called **sensitivity** since the larger this coefficient the greater the stimulus influence. The slope B is a tangent of the angle α.

This model assumes that the transfer function passes through zero value of the input stimulus s. In many practical cases it is just difficult or impossible to test a sensor at a zero input.

For example, a temperature sensor used on a Kelvin scale cannot be tested at the absolute zero (-273.15 °C).

2.1 Mathematical models

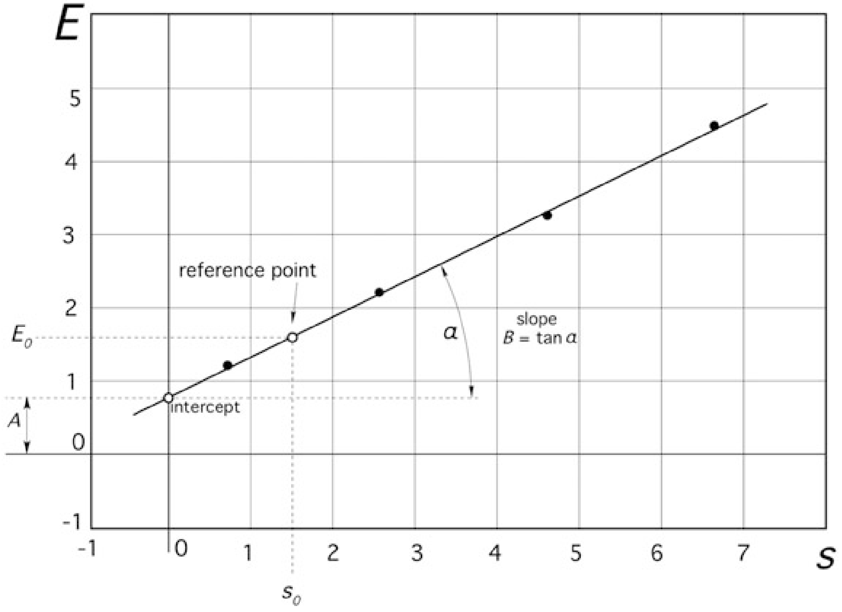
Thus, in many linear or quasilinear sensors it may be desirable to reference the sensor not to the zero input but rather to some more practical input reference value s_0.

If the sensor response is E_0 for some known input stimulus s_0, previous equation can be rewritten as:

$$E = E_0 + B(s - s_0)$$

The reference point has coordinates s_0 and E_0. The inverse linear transfer function for computing the input stimulus from the output E is:

$$s = \frac{E - E_0}{B} + s_0$$



2.1 Mathematical models

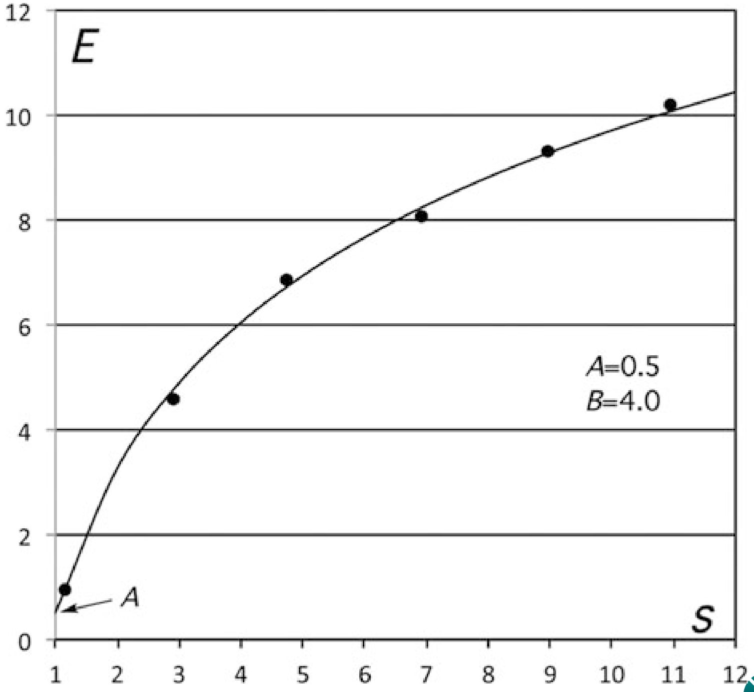
In the real world, nonlinearities are often present especially for a broad input range of the stimuli.

A nonlinear transfer function can be approximated by a nonlinear mathematical function, as logarithmic, exponential and power function.

The **logarithmic approximation function** and the corresponding inverse function (which is exponential) are respectively:

$$E = A + B \ln(s)$$
$$s = e^{\frac{E-A}{B}}$$

where A and B are the fixed parameters.



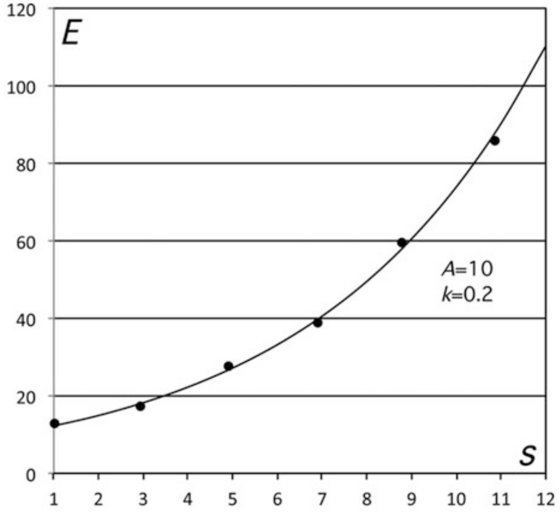
2.1 Mathematical models

The **exponential function** and its inverse (which is logarithmic) are given by:

$$E = Ae^{ks}$$

$$s = \frac{1}{k}\ln\left(\frac{E}{A}\right)$$

Where A and k are the fixed parameters.

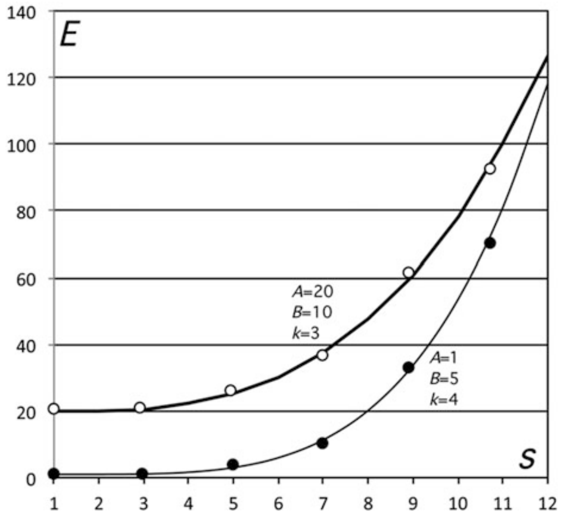


The **power function** and its inverse are expressed as:

$$E = A + Bs^k$$

$$s = \sqrt[k]{\frac{E - A}{B}}$$

Where A and B are the fixed parameters, while k is the power factor.



2.1 Mathematical models

All the above three nonlinear approximations possess a **small number of parameters** that shall be determined during calibration.

A small number of parameters makes them rather convenient, provided that they can fit response of a particular sensor.

It is always useful to have as small a number of parameters as possible, not the least for the sake of lowering cost of the sensor calibration.

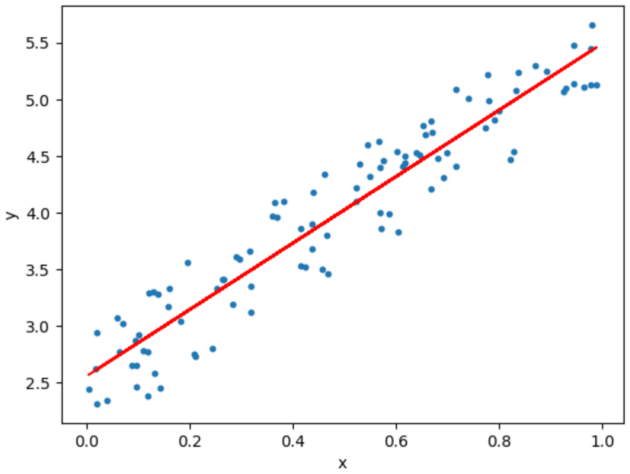
The fewer parameters, the smaller the number of the measurements to be made during calibration.

2.1 Mathematical models

If measurements of the input stimuli during calibration cannot be made consistently with high accuracy and large random errors are expected, the minimal number of measurements will not yield a sufficient accuracy.

To cope with random errors in the calibration process, a **method of least squares** could be employed to find the slope and intercept.

For linear transfer function, one can measure multiple (k) output values E at the input values s over a substantially broad range, preferably over the entire sensor span and use the formulas for a linear regression to determine intercept A and slope B of the best-fitting straight line:



$$A = \frac{\sum E \sum s^2 - \sum s \sum s \cdot E}{k \sum s^2 - (\sum s)^2}; B = \frac{k \sum sE - \sum s \sum E}{k \sum s^2 - (\sum s)^2}$$

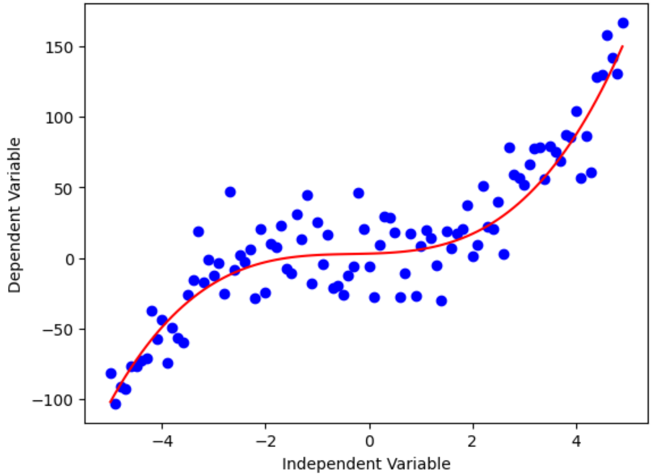
2.1 Mathematical models

A sensor may have such a transfer function that none of the above basic functional approximations would fit sufficiently well.

In this case, nonlinear functions can be approximated using a **polynomial approximation**, that is, a power series.

In some cases, especially when more accuracy is required, the higher order polynomials should be considered because the higher the order of a polynomial the better the fit.

Still, even a second-order polynomial often may yield a fit of sufficient accuracy when applied to a relatively narrow range of the input stimuli and the transfer function is monotonic (no ups and downs).



2.1 Mathematical models

For a nonlinear transfer function, sensitivity is not a fixed number, as would be the case in a linear transfer function.

A nonlinear transfer function exhibits different sensitivities at different points in intervals of stimuli.

In the case of nonlinear transfer functions, sensitivity is defined as a first derivative of the transfer function at the particular stimulus s_i:

$$b_i(s_i) = \frac{dE(s_i)}{ds} = \frac{\Delta E_i}{\Delta s_i}$$

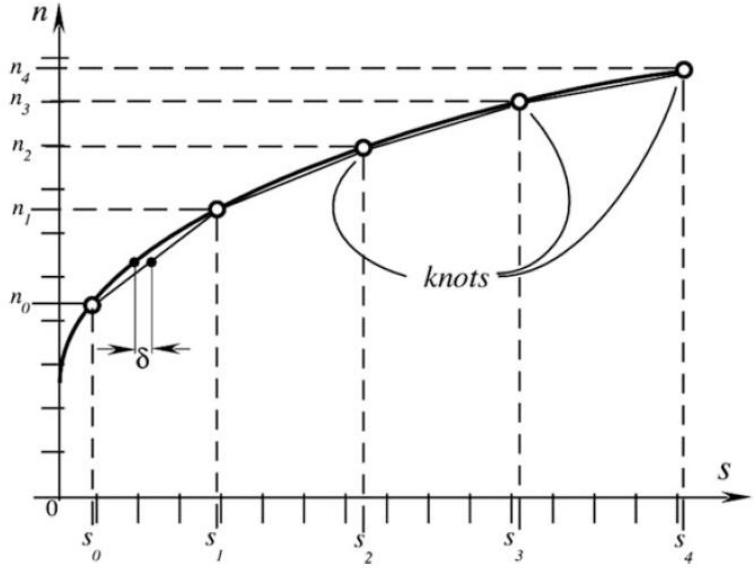
where, Δs_i is a small increment of the input stimulus and ΔE_i is the corresponding change in the sensor output E.

2.1 Mathematical models

For nonlinear functions, linear piecewise approximation is a powerful method. The idea behind it is to break up a nonlinear transfer function of any shape into sections and consider each such section being linear.

Curved segments between the sample points (knots) demarcating the sections are replaced with straight-line segments, thus greatly simplifying behavior of the function between the knots. This can also be seen as a polygonal approximation of the original nonlinear function.

The knots do not need to be equally spaced. They should be closer to each other where nonlinearity is high and farther apart where nonlinearity is small.

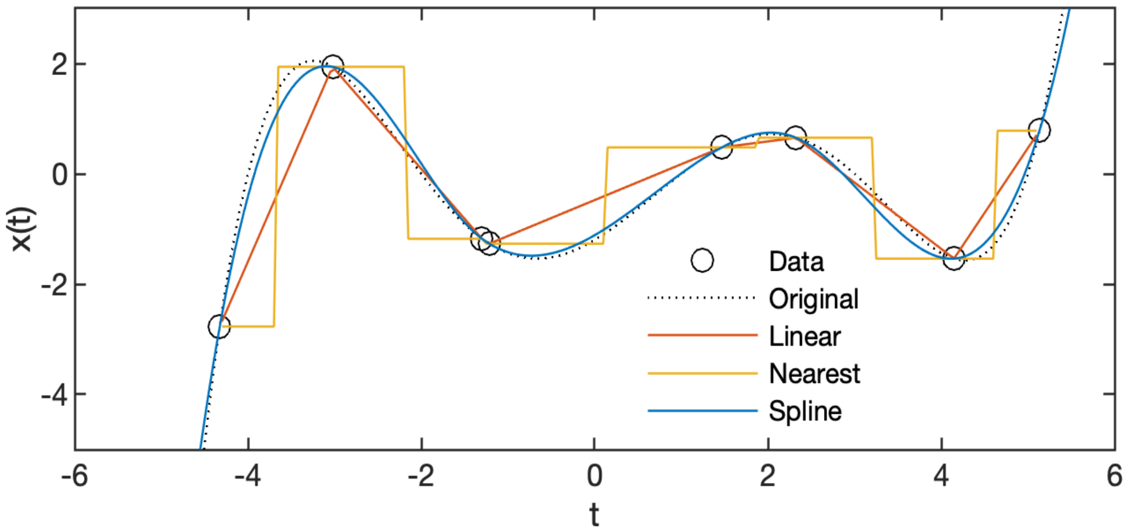


2.1 Mathematical models

Approximations by higher order polynomials (third order and higher) have some disadvantages as the selected points at one side of the curve make strong influence on the remote parts of the curve.

This deficiency is resolved by the spline method of approximation. In a similar way to a linear piecewise interpolation, the spline method is using different third-order polynomial interpolations between the selected knots.

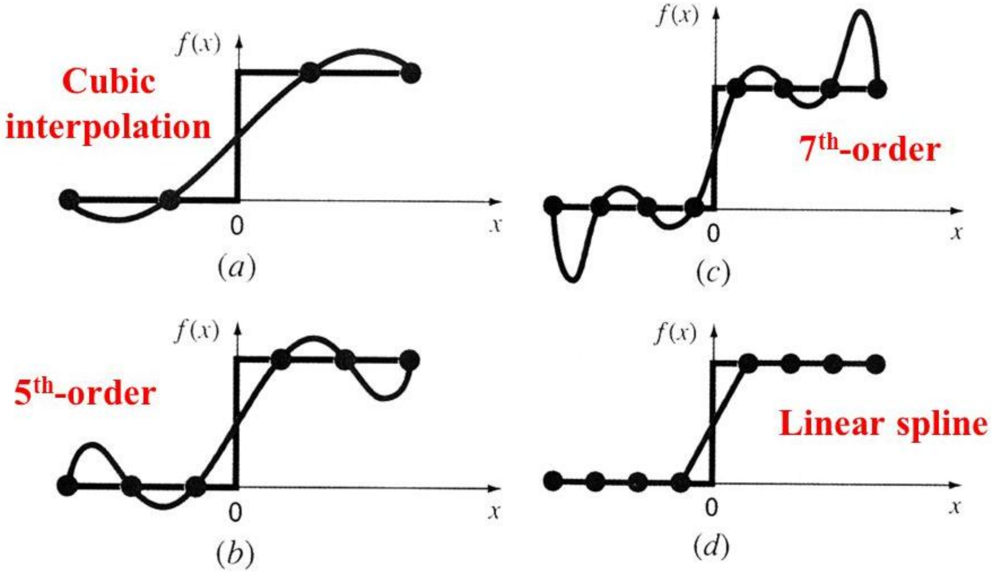
The spline is a unique piecewise polynomial such that its point values and its first two derivatives are continuous at the given n points (knots).



2.1 Mathematical models

The spline interpolation can utilize polynomials of different degrees, yet the most popular being cubic (third order) polynomials.

Curvature of a line at each point is defined by the second derivative. This derivative should be computed at each knot.



2.1 Mathematical models

A sensor transfer function may depend on more than one input variable. That is, the sensor's output may be a function of several stimuli.

One example is a humidity sensor whose output depends on two input variables: relative humidity and temperature.

Another example is the transfer function of a thermal radiation (infrared) sensor. This function has two arguments, namely two temperatures: Tb, the absolute temperature of an object of measurement and Ts, the absolute temperature of the sensing element. Thus, the sensor's output voltage V is proportional to a difference of the fourth-order parabolas:

$$V = G\left(T_b^4 - T_s^4\right)$$

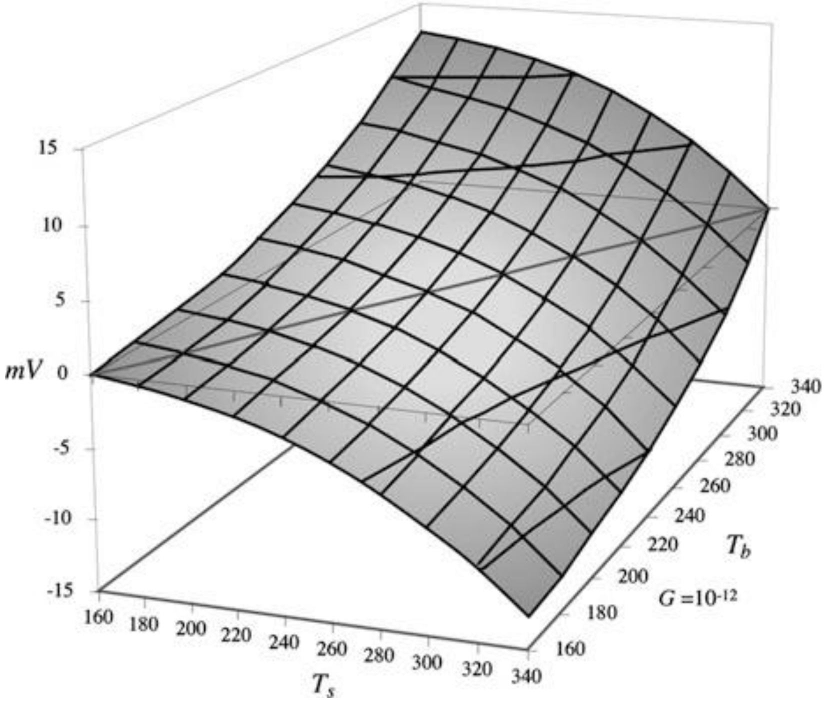
where G is a constant.

2.1 Mathematical models

The relationship between the object's temperature T_B and the output voltage V is not only nonlinear but also in a nonlinear way depends on the sensing element surface temperature T_s.

To determine the sensitivity of the sensor with respect to the object's temperature, a partial derivative will be calculated as:

$$b = \left.\frac{dV}{dT_b}\right]_{T_s} = 4GT_b^3$$



2.2 Calibration process

The operation required to retrieve the transfer function, and thus correlate the stimulus and the response of the sensor, is called **calibration**

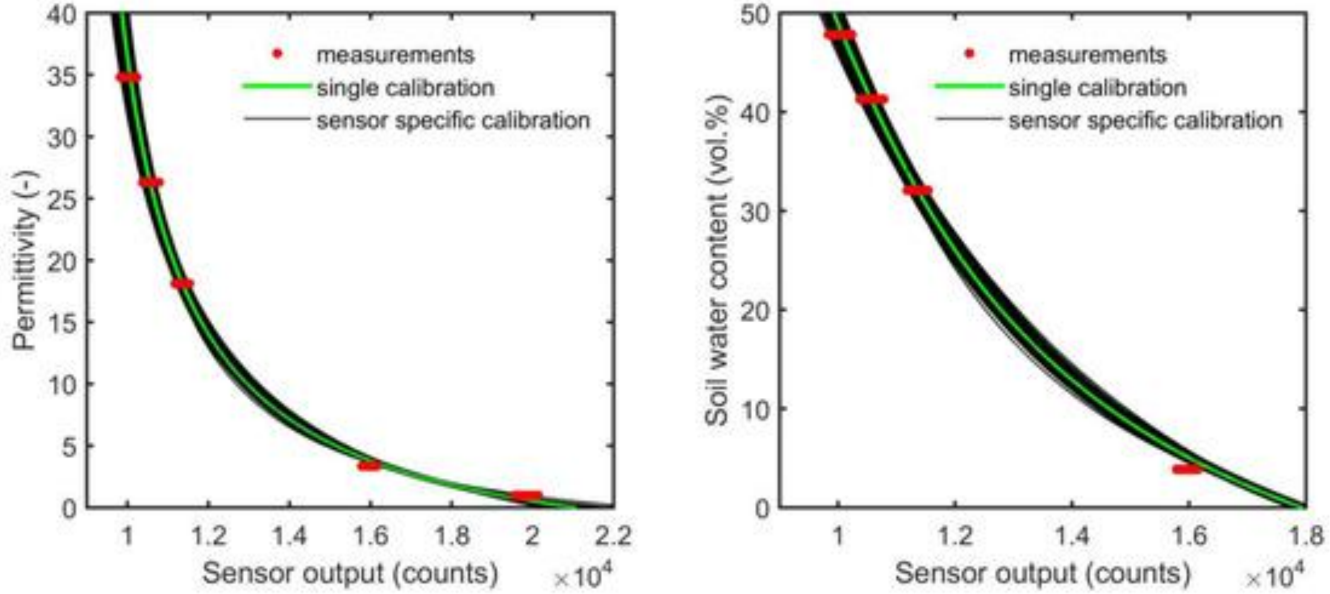
A calibration requires application of several precisely known stimuli and reading the corresponding sensor responses. These are called the **calibration points** whose input–output values are the point coordinates. In some lucky instances only one pair is required, while typically 2–5 calibration points are needed to characterize a transfer function with a higher accuracy. After the unique transfer function is established, any point in between the calibration points can be determined.

The reference source should be well maintained and periodically checked against other established references.

2.2 Calibration process

Before calibration, either a mathematical model of the transfer function has to be known or a good approximation of the sensor's response over the entire span shall be found. In a great majority of cases, such functions are smooth and monotonic.

Very rarely they contain singularities and if they do, such singularities are the useful phenomena that are employed for sensing.



2.2 Calibration process

Calibration of a sensor can be done in several possible ways:

- **Modifying the transfer function or its approximation to fit the experimental data.** This involves computation of the coefficients (parameters) for the selected transfer function equation. After the parameters are found, the transfer function becomes unique for that particular sensor. The function can be used for computing the input stimuli from any sensor response within the range. Every calibrated sensor will have its own set of the unique parameters. The sensor is not modified.

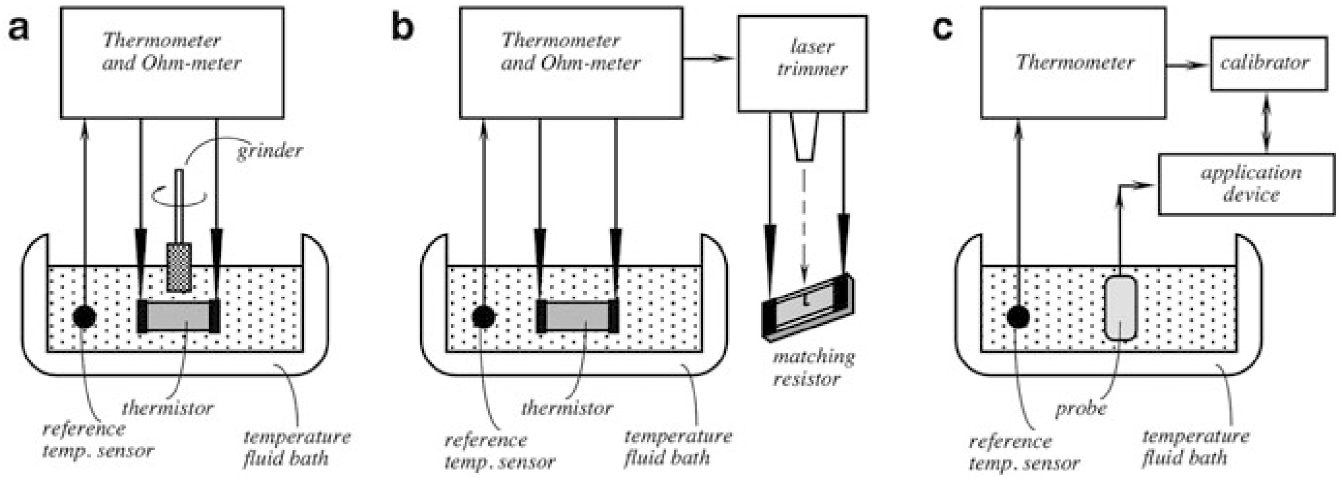
- **Adjustment of the data acquisition system to modify its output by making the outputs signal to fit into a normalized or “ideal” transfer function.** An example is a scaling and shifting the acquired data (modifying the system gain and offset). The sensor is not modified.

2.2 Calibration process

- **Creating the sensor-specific reference device with the matching properties at particular calibrating points**. This unique reference is used by the data acquisition system to compensate for the sensor's inaccuracy. *The sensor is not modified*.
- **Modification (trimming) the sensor's properties** to fit the predetermined transfer function, thus the *sensor itself is modified*.

Example: calibrating a thermistor (temperature sensitive resistor)

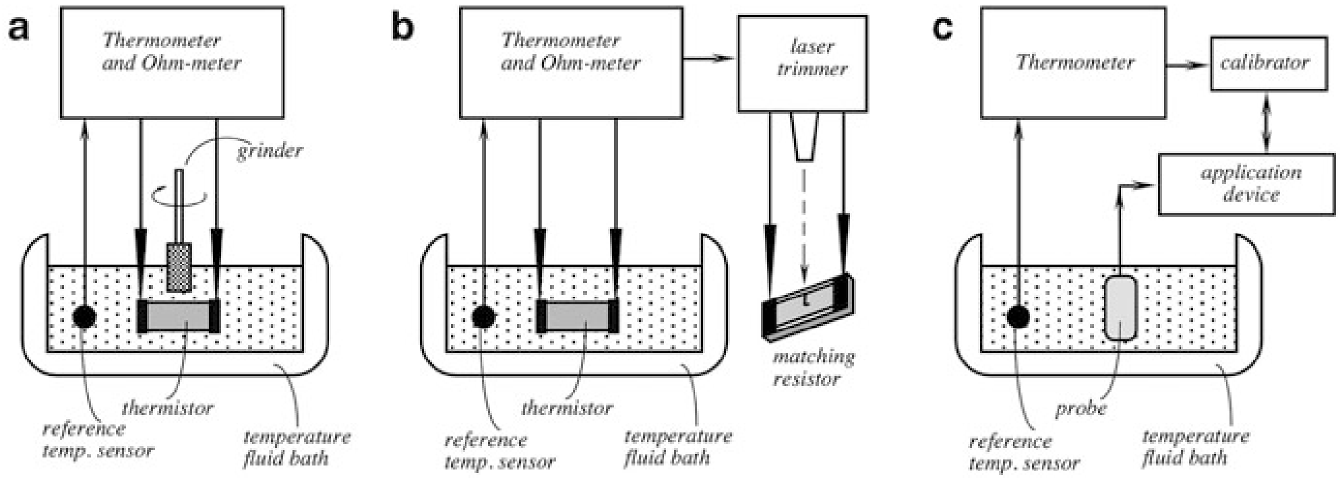
2.2 Calibration process



a) The thermistor is immersed into a stirred liquid bath with a precisely controlled and monitored temperature. A grinder mechanically removes some material from the thermistor body to modify its dimensions. Reduction in dimensions leads to increase in the thermistor electrical resistance at the selected bath temperature.

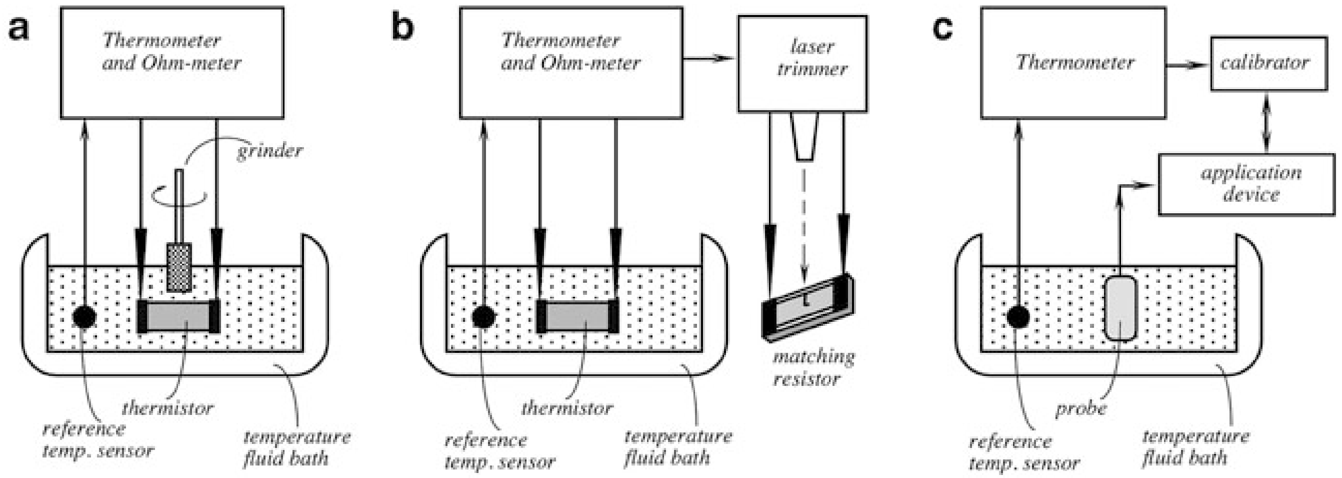
When the thermistor's resistance matches a predetermined value of the "ideal" resistance, the grinding stops and the calibration is finished.

2.2 Calibration process



b) The thermistor is not modified but just measured at a selected reference temperature. The measurement provides a number that is used for selecting a conventional (temperature stable) matching resistor as a unique reference. That resistor is for use in the interface scaling circuit. That individually matched pair thermistor–resistor is used in the measurement circuit, for example, in a Wheatstone bridge.

2.2 Calibration process

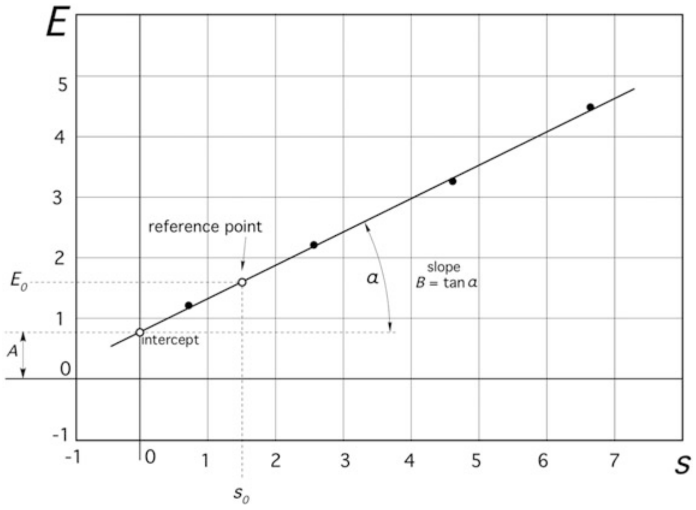


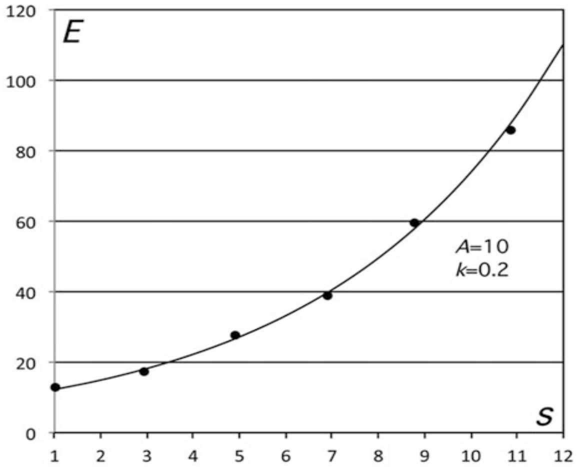
c) Multiple calibrating points at different temperatures are generated. The liquid bath is sequentially set at two, three, or four different temperatures and the thermistor under calibration produces the corresponding responses, that are used by the calibrating device to generate the appropriate parameters for the inverse transfer function that will be stored in the application device (e.g. a thermometer).

2.3 Computation of parameters

If a transfer function is linear then calibration should determine constants A and B, slope and offset.
If it is exponential the constants A and k should be determined, and so on.

To calculate parameters of a **linear transfer function**, at least two data points defined by two calibrating input–output pairs are required. For a simple linear transfer function two points are required to define a straight line, thus a two-point calibration can be performed.





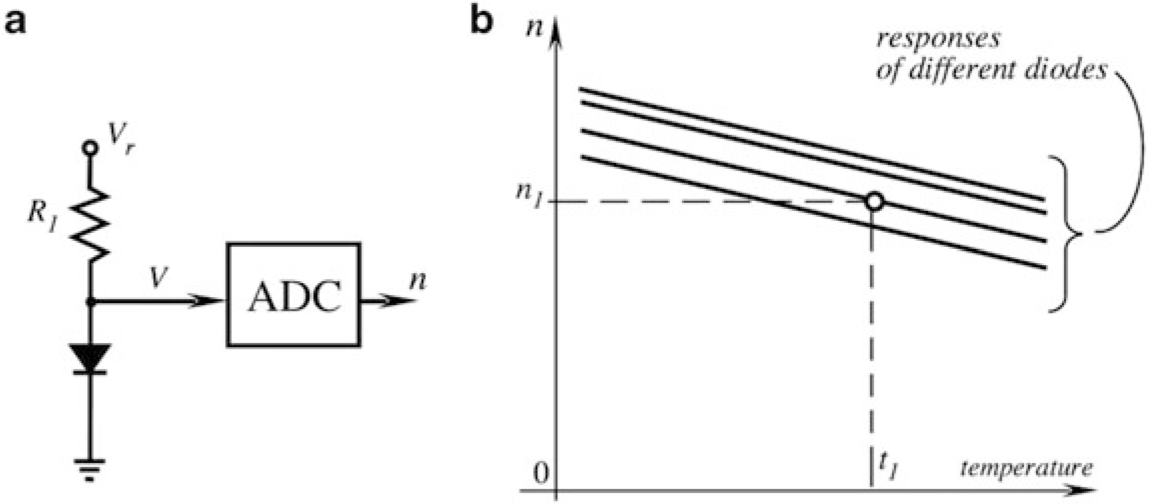
2.3 Computation of parameters

Example: temperature sensor based on forward biased semiconductor p–n junction.

The system is characterized by a linear transfer function with temperature *t* being the input stimulus and the ADC count *n* from the interface circuit is the output:

$$n = n_1 + B(t - t_1)$$

To fully define the line, the sensor shall be subjected to two calibrating temperatures (t_1 and t_2) for which two corresponding output counts (n_1 and n_2) will be registered



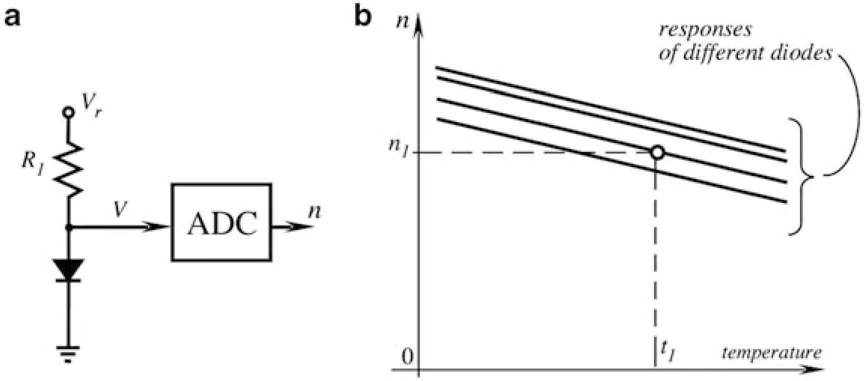
2.3 Computation of parameters

After subjecting the sensor to the second calibrating temperature t_2, we receive the digital counts for the second calibrating point. The count is:

$$n_2 = n_1 + B(t_2 - t_1)$$

from which the sensitivity (slope) is computed as:

$$B = \frac{n_2 - n_1}{t_2 - t_1}$$



The sensitivity (slope) B is in count/degree.

The parameters found from calibration are unique for the particular sensor and must be stored in the measurement system to which that particular sensor is connected. For another similar sensor, these parameters will be different (except t_1, if all sensors are calibrated at exactly the same temperature).

2.3 Computation of parameters

After calibration is done, any temperature within the operating range can be computed from the ADC output count n by use of the inverse transfer function

$$t = t_1 + \frac{n - n_1}{B}$$

For **nonlinear transfer functions**, calibration at one data point may be sufficient only in some rare cases when other parameters are already known, but often two and more input–output calibrating pairs would be required.

For example, when a second or a third degree polynomial transfer functions are employed, respectively three and four calibrating pairs are required.